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SOME FLOWS IN A GRAVITY FIELD SATISFYING THE EXACT FREE SURFACE CONDITION

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Introduction

Although the earliest students of hydrodynamics were strongly interested in such natural phenomena as fountains and water waves, progress in attaining a mathematical description of these phenomena has been relatively slow. It was not, in fact, until the nineteenth century that the mathematical framework of hydrodynamics was established on a firm basis. Since then, knowledge in this field has accumulated -- along with that in the other major sciences -- at an accelerating rate. Yet there remain -- as in the other sciences -- important questions that are still unanswered. For example, we may ask what are the precise shape and velocity of a simple surface wave on the ocean? Although these entities can be approximated to a high degree of accuracy, there exist as yet no formulas which describe them exactly.

The basic mathematical difficulty lies in the treatment of the free surface boundary of the fluid. In the usual problem involving a boundary, the boundary is not free, but is instead fixed and its location is known. The customary approach to this problem then involves the setting up of a mathematical expression which states that the fluid must not pass through the boundary. However, in a hydrodynamical problem involving a free surface boundary, the character of the boundary value problem assumes a special nature. On the boundary -- the free surface -- one prescribes two conditions, namely, that the free surface is not violated by the fluid, and, further, that the pressure is constant there. However, until we have solved the problem, we do not know the shape of the free surface. Thus we must solve a boundary value problem with an unknown boundary.

Historically, the early investigations of surface waves used linearization -- a technique in which, essentially, the free surface is assumed to be

flat for the purpose of prescribing boundary conditions. As long as the waves are very small, this approximation is justified. The next approach was to use the exact equations, but to neglect all terms of order higher than the second. The applicability was thus extended to include larger waves. Outstanding among the contributions to the theory of waves of finite amplitude are those of Stokes [10] and Rayleigh [8]. It is hardly necessary to go into the history of this theory, since excellent treatments can be found in readily available works such as Lamb [3]. A less technical treatment is given by Sverdrup, et al [11]. Rossby [9] has shown that the system of deep water waves of finite amplitude given by Stokes is only one of an infinite number of possible irrotational wave systems. Gernstner [2] dropped the requirement that the motion be irrotational and was able to obtain an exact solution to the remaining equations -- those of continuity and constant surface pressure -- which showed surface waves over deep water to have a trochoidal profile. His work will be mentioned again in Chapter II.

In the exact theory for irrotational waves -- the theory with which this paper is concerned -- Levi-Civita [5] has shown that finite waves of permanent form do exist. It might appear, especially to one who has witnessed the evidence, that an existence proof for ocean waves is unnecessary. However, this is not the case, for in reality it is only in the mathematical treatment that true permanent waves actually exist. We neglect viscosity and hence the frictional forces that in nature tend to cause a wave, in time, to lose its force. Thus in deviating slightly from physical reality, we make it necessary to prove the existence of a solution to our problem. The existence proof of Levi-Civita yielded a fair approximation to the shape of the wave. There has been some interest in the solitary wave problem also:

the question is whether or not a single wave of permanent form can exist on the ocean, inasmuch as a sudden disturbance usually sets up a train of waves instead of a single wave. The question has been answered in the affirmative by Lavrentieff [4], but he did not give the exact shape of the wave.*/ This problem is mentioned because of the fact that the methods of this paper could be applicable to it.

A fresh approach to free-surface problems in hydrodynamics has been made possible recently by the introduction of a new mathematical tool [6]. This new procedure allows us to generate free-surface flows at will, and it forms the fundamental concept upon which this paper is based. The method is an extension to the study of non-gravity flows of classical investigations based on the theory of functions of a complex variable. As is pointed out in most textbooks on the subject, any analytic function can be interpreted as the solution to a two-dimensional hydrodynamical flow problem. Our analysis is somewhat reversed from normal procedure, however, in that one must first decide on a function, and then determine what hydrodynamical flow problem it solves. Of course, more than pure guesswork is involved. Having a particular flow problem in mind, one proceeds, as an analytical detective, to uncover certain clues which will enable him to choose the proper function. In this paper, we thus construct several examples of free-surface flows using a formula suggested by Hans Lewy [6]. He found that whereas any analytic function represents a hydrodynamical flow, if we restrict the functional relationship in a certain way, the flow then becomes a free-surface flow in a gravity field.

As is often the case with mathematical investigations of physical

*/ This theorem has also been proved recently by Friedrichs and Hyers.

phenomena, one must make some plausible assumptions in order to simplify the mathematics. In this paper we make the usual assumptions concerning the fluid, namely, that it is incompressible and is without viscosity. Further, our investigations will deal with phenomena which are representable in two dimensions, and which can be generated from rest -- that is, which are irrotational. Bernoulli's law will be used in its exact form.

Now that our medium is defined we may state Lewy's discovery explicitly. He found that if $\lambda(\zeta)$ is an arbitrary analytic function of ζ which is real on the real ζ-axis, then the mapping

$$z(\zeta) = -i\,\lambda(\zeta) + \int \sqrt{\frac{1}{2\lambda(\zeta)} - \left(\frac{d\lambda(\zeta)}{d\zeta}\right)^2}\, d\zeta \tag{1}$$

represents a steady free-surface gravity flow in the z-plane, the free surface being the image of a segment of the real ζ-axis along which the inequality

$$\frac{1}{2\lambda} - \left(\frac{d\lambda}{d\zeta}\right)^2 \geq 0$$

holds. In Chapter I, this relation will be developed in a manner different from that used by Lewy. The existence of λ for such a free-surface flow and the proof of the analyticity of the free surface will also be given. It should be pointed out that this work was done by Lewy in a more general manner than that presented here. It is hoped that the loss in generality is compensated for by a gain in simplicity.

The major contribution of this paper lies in the application of Lewy's theory. Up until this time there have been no important free-surface flows in a gravity field represented in closed form. The solutions have, in the past, been either approximations (linearized theory) or the first few terms of an infinite series solution. With Lewy's method one obtains a solution

which is in closed form. The major difficulty with the method lies in trying to pick the few significant functions $\lambda(\zeta)$ out of the infinite number of possible functions. The functions which will yield the solutions to three outstanding problems of surface waves remain as yet unknown. These are the problems of periodic waves over a flat bottom of finite depth, periodic waves over an ocean of infinite depth, and the solitary wave. It is hoped that this paper will be of some aid in the quest for the solutions to these problems.

In Chapter II an example of a free-surface wave will be given. It is of trochoidal profile and travels over a smooth, but not a flat, bottom. Chapter III contains a limiting case of the flow of Chapter II where the trochoid becomes a cycloid. It is shown that when this occurs, the character of the flow changes completely in that the periodic nature of the flow is destroyed. Some general properties of deep-water waves are studied in Chapter IV with respect to their effect on the choice of the arbitrary function $\lambda(\zeta)$. An example of surface waves growing out of these considerations is given. In Chapter V we leave the realm of surface waves and examine two flows of a non-periodic nature. One is a fountain-like flow which yields many interpretations. It is especially interesting because it provides a first example of how a stagnation point on the surface need not necessarily occur at an angle of $2\pi/3$. The second example might be interpreted as the skimming action of a vertical board being drawn rapidly along the surface of a deep body of fluid.

It should be mentioned that the attack on the problems of free-surface phenomena received impetus during the war because of its military importance, the major interest centering around ocean waves. In order to prepare

adequately for an amphibious assault, one needed a means of forecasting wind waves and swell. A part of this study required a knowledge of shapes and velocities of the waves when, having been created by storm, they moved on through the undisturbed sea. A knowledge of the shape and velocity of ocean waves is also important to those making a study of ship stabilization, since the forces on the hull of a ship at sea are dependent on them. Under certain conditions, one can see ocean waves quite clearly from the air. To the airman, the exact shape of the waves is not too important, but the velocities might well concern him. A ship at sea forms directly behind it a train of waves called the transverse wake and these waves move with the same velocity as the ship. From the air one can measure the wavelength of this transverse wake and consequently, knowing the velocity of a wave as a function of its wavelength, one is able to determine the speed of the ship. Also, the knowledge of wavelength as a function of velocity and depth can be applied to determine, from aerial photographs, the depth of a bay, the slope of a beach, and other geographical data.

I would like to express my great appreciation for the help given by Professor Hans Lewy in my early work on the applications of his theory. And, of course, the guidance and inspiration received from my advisor Professor Paul Garabedian is without measure.

Chapter I

As was mentioned in the introduction, we are investigating steady, two-dimensional, irrotational, free-surface flows of an incompressible, inviscid fluid in a gravity field. From the irrotationality, we know that there exists a velocity potential φ such that the velocity, $\vec{q}$, is the gradient of φ. For steady incompressible flows the equation of continuity states that the divergence of $\vec{q}$ is zero; therefore φ satisfies Laplaces's equation $\nabla^2 \varphi = 0$. Finally, the dynamical equations yield Bernoulli's law, which, in our case, takes the form

(2) $$1/2\ q^2 + gy + \frac{P}{\rho} = \text{constant},$$

where q, without the arrow, is the magnitude of the velocity $\vec{q}$, g is the acceleration due to gravity, y is the vertical coordinate (positive upwards), P is the pressure at any point of the fluid, and ρ is the density of the fluid. If we introduce the stream function ψ, we may describe a two-dimensional flow by the complex potential

$$\zeta(z) = \varphi(z) + i\psi(z).$$

Then

$$\frac{d\zeta}{dz} = \varphi_x + i\psi_x = \varphi_x - i\varphi_y = q_1 - i q_2,$$

where q_1 and q_2 are the x and y components of velocity, respectively, whence

$$\vec{q} = \overline{\left(\frac{d\zeta}{dz}\right)},$$

the bar denoting the complex conjugate. The above development is familiar from textbooks on hydrodynamics or complex variables and consequently we

omit further detail here and proceed to develop Lewy's theory.

Physically a stream line is a line across which no fluid will flow -- thus any stream line is a possible boundary for the flow. If we want one of our stream lines to be a free surface boundary, then we must, in addition, require that the pressure be constant all along this stream line. Conversely, if along some segment of a stream line the pressure turns out to be constant, then it is a possible free surface. For the best physical example to illustrate this constant-pressure criterion, we again turn to the ocean. The constant pressure along the free surface is just equal to the atmospheric pressure of the air. Since the inertia of air is very small compared to that of water, the surface of the water may easily displace the air and so is not impeded by the air during a change in shape. That is to say, the surface is <u>free</u> to change shape. This is an important factor in progessive waves, for if they were required to do much work in displacing the air, they would soon lose their force. Of course, the constant-pressure criterion is again a mathematical idealization, for air does have its effect on surface waves, especially when the air is itself in motion. Indeed, most ocean waves owe their existence to the forces created by air in rapid motion.

In any flow represented by a complex potential, a stream line is characterized as the image of a line $\mathrm{Im}(\zeta)$ = constant. One of the stream lines in the z-plane is hence the image of the line $\mathrm{Im}(\zeta) = 0$, and it is this line that we will select for the free surface. We now turn our attention entirely to this free-surface stream line. It may be that only part of the real ζ-axis will satisfy the free-surface criterion of constant pressure. At any rate, whether it be all or only a part of the ζ-axis that is the image of the free surface, we will call it the free-surface segment of the real ζ-axis.

Along this free-surface segment, or, more properly, along its image in the z-plane, the pressure P is constant. Since the fluid is incompressible, the density ρ is a constant throughout the fluid and, therefore, in particular, on the free surface. We may, therefore, write Bernoulli's law in the form

$$(3) \qquad 1/2\, q^2 + gy = -\frac{P}{\rho} + \text{constant},$$

where everything on the right-hand side is constant. We are at liberty to choose the location of the origin in the z-plane and can do so in such a way that the constant gk obtained from replacing y by some y* + k just cancels the two constants on the right-hand side of equation (3). This means, in particular, that the zero-velocity level of the fluid coincides with the real z-axis when we are on a free surface. Notice that if we are not on a free surface, there is no reason to expect zero velocity on the real z-axis, since then the pressure P is not necessarily constant. A saving in effort is achieved if we further set the acceleration of gravity equ l to 1. Physically this means we are choosing our unit of length to be equal to approximately 32 feet, provided time is still measured in seconds. Thus, on the free surface, Bernoulli's law takes the simple form

$$(4) \qquad 1/2 \left| \frac{d\zeta}{dz} \right|^2 + y = 0 .$$

This can be inverted and rewritten as

$$(5) \qquad \frac{dz}{d\zeta} \frac{d\bar{z}}{d\bar{\zeta}} + \frac{1}{2y} = 0 .$$

This equation is true only on the free surface, with dz and $d\zeta$ measured along the free surface. But the free surface is the image of $\mathrm{Im}(\zeta) = 0$, and therefore we may replace $d\bar{\zeta}$ by $d\zeta$, since both are equal to $d\varphi$ and are

real there. Hence equation (5) becomes

$$\frac{dz}{d\zeta}\frac{d\bar{z}}{d\zeta} + \frac{1}{2y} = 0 .$$

Since $z - \bar{z} = 2iy$, we have $\bar{z} = z - 2iy$, so that

$$\frac{d\bar{z}}{d\zeta} = \frac{dz}{d\zeta} - 2i\frac{dy}{d\zeta} ,$$

where $\frac{dy}{d\zeta} = \frac{dy}{d\varphi}$ has meaning, since the free surface can be represented parametrically in the form $z = z(\varphi) = x(\varphi) + iy(\varphi)$. We note that from equation (4), the inversion $z = z(\zeta)$ is always possible if the free surface is bounded away from the real z-axis. If we let a prime ' designate differentiation with respect to ζ, we have finally

$$z'(z' - 2iy') + \frac{1}{2y} = 0 . \tag{6}$$

We now come to the essence of Lewy's idea. Equation (6) is an equation, involving a real quantity y, that must hold on the free surface. But suppose we consider it as an equation in the complex domain involving a complex function y. Then its solution would be an analytic function (under proper restrictions) and hence would represent a hydrodynamical flow. If this complex quantity became real along the stream line $\mathrm{Im}(\zeta) = 0$, then by virtue of the derivation of equation (6), that stream line would be a free stream line. To accomplish this, we replace $-y$ by an analytic function of ζ, $\lambda(\zeta)$, which is real and positive along some segment of the real ζ-axis. If we insert this function into equation (6) we have

$$z'(z' + 2i\lambda') - \frac{1}{2\lambda} = 0 . \tag{7}$$

Since $\lambda(\zeta)$ is analytic, equation (7) is a non-linear differential equation

which connects z with ζ in an analytic way. Consequently, it determines a hydrodynamical flow. Furthermore, the image of a segment of the real ζ-axis is a stream line along which pressure is constant by virtue of the derivation of equation (7), provided $\lambda = -y$ on the stream line -- a fact we establish immediately. Hence this stream line is a free-surface stream line.

The differential equation (7) may be solved for z in terms of an integral,

$$z = -i\lambda + \int \sqrt{\frac{1}{2\lambda} - \lambda'^2}\, d\zeta. \tag{8}$$

It is clear from this equation that $\lambda(\zeta)$ is equal to -y on the free surface, provided the integral itself is real on the free surface. Consequently, we must require

$$\frac{1}{2\lambda} - \lambda'^2 \geq 0 \tag{9}$$

along the free-surface segment. We call this the free-surface condition.

We now have a method of constructing free-surface flows. Using equation (8) we may generate a free-surface flow by inserting any function $\lambda(\zeta)$ which is 1) analytic, 2) greater than zero (hence real) on some segment of the real ζ-axis, and 3) such that $1/2\lambda \geq \lambda'^2$ on this segment. Recall that in the classical theory, the arbitrary analytic function itself determines the flow, whereas here the arbitrary analytic function appears as a parameter and as a consequence exerts a more indirect influence on the resulting flow.

It is also possible to make a converse statement. For any free-surface flow, there exists a unique function $\lambda(\zeta)$ which will generate the flow in the above sense. Furthermore, any free-surface flow may be continued across the free surface and as a consequence, the free surface must be an analytic

curve. Before going ahead with the proof of these statements, we will demonstrate the necessary existence and uniqueness theorems for first-order ordinary differential equations in the complex domain.

The existence and uniqueness theorems needed here differ slightly from those usually given in the textbooks. Often the initial values are taken at the centers of the domains under consideration. We will need to assign an initial value to the independent variable which is on the boundary of the domain. Further, the usual proof just shows that the solution exists and is analytic within the domain under consideration. We will need both existence and uniqueness on the boundary of the domain, and we will prove that the solution is also continuous on the boundary.

Consider the differential equation

$$\frac{d\lambda}{d\zeta} = f(\zeta,\lambda) \tag{10}$$

together with the following domains. Let D_1 be an open circular region of the λ-plane of radius r and center λ_0. Let D_2 be an open simply-connected region of the ζ-plane which may be of any configuration, provided that any two points belonging to D_2 can be joined by a path of finite length contained in the closure of D_2. Choose the radius r of D_1 and the domain D_2 so that $f(\lambda,\zeta)$ is regular and single valued for both variables ranging within their respective domains, is continuous in both variables within and on the boundaries of these domains, and satisfies the Lipshitz condition

$$\left|f(\zeta,\lambda_1) - f(\zeta,\lambda_2)\right| < K\left|\lambda_1 - \lambda_2\right|$$

for λ_1 and λ_2 within or on the boundary of D_1 and ζ belonging to the closure of D_2. Further, let M be the maximum value of $\left|f(\zeta,\lambda)\right|$ for y and λ belonging

to the closures of their respective domains. The existence of the maximum is assured by the continuity of the function $f(\zeta, \lambda)$. Now let ζ_0 belong to the closure of the domain D_2, where we specifically allow values on the boundary of D_2. If we denote the greatest length of path necessary*/ to join ζ_0 to any point ζ that belongs to the closure of D_2 by S, and if $S < \frac{r}{M}$, then we may proceed with the analysis. If, however, $S > \frac{r}{M}$, then we alter D_2 by removing those parts farthest away (by path) from ζ_0 until the inequality $S < \frac{r}{M}$ holds. Call this altered domain the domain D, and set $D = D_2$ when no alterations are necessary. Then the differential equation (10) admits a unique solution $\lambda(\zeta)$ with $\lambda_0 = \lambda(\zeta_0)$ which is analytic within D and continuous in the closure of D.

Suppose for a moment that a solution to the differential equation is known which has as initial value λ_0 for $\zeta = \zeta_0$. This solution would satisfy the relation

$$\lambda(\zeta) = \lambda_0 + \int_{\zeta_0}^{\zeta} f(t, \lambda(t))dt.$$

If the function $\lambda(\zeta)$ is unknown, then this relation becomes an integral equation which may be solved by the method of successive approximations in the following manner.

Consider the sequence of functions

*/Although it is not really necessary, we will assume for sake of clarity that a geodesic exists for all pairs of points belonging to the closure of D_2. We say geodesic (path of shortest length) because it is possible that two points could not be joined by a straight line lying in the closure of the domain D_2. In that case parts of the boundary must be used together with straight-line segments.

$$\lambda_1(\zeta) = \lambda_0 + \int_{\zeta_0}^{\zeta} f(t, \lambda_0)dt,$$

$$\lambda_2(\zeta) = \lambda_0 + \int_{\zeta_0}^{\zeta} f(t, \lambda_1)dt,$$

$$\lambda_n(\zeta) = \lambda_0 + \int_{\zeta_0}^{\zeta} f(t, \lambda_{n-1})dt,$$

where each of the integrals is a line integral from ζ_0 to some point ζ in D along a path each point of which belongs to the closure of D. Since λ_0 is a constant within D_1, $f(t, \lambda_0)$ is a regular, single-valued function of t for t within D, and is continuous in the closed domain D. By Cauchy's integral theorem the value of λ_1 is independent of the path of integration and, further, by Morera's theorem, λ_1 is regular and single valued within D. The continuity of λ_1 for all ζ belonging to the closure of D follows from the fact that it is an integral. Note that the path of integration may include part of the boundary of D, for Cauchy's theorem requires only analyticity within the contour and continuity on the contour. The endpoint ζ may be on the boundary, and the value of λ_1 on the boundary is quite independent of our path; that is, we may reach this boundary point by integration along part of the boundary or by approaching it from within D.

Turning our attention to λ_2, we see that we will be able to attribute the same properties to λ_2 as we have to λ_1 as soon as we have shown $f(t, \lambda_1)$ to be regular and single valued within D and continuous for ζ belonging to the closure of D. To do this we must first show that λ_1 is within D_1. Now

$$\lambda_1 - \lambda_0 = \int_{\zeta_0}^{\zeta} f(t, \lambda_0)dt,$$

where the value on the left is independent of the path of integration. Consequently, for purposes of obtaining a good estimate, we may choose the geodesic joining ζ_0 to ζ as our path of integration. We have

$$\left|\lambda_1 - \lambda_0\right| \leq \left|\int_{\zeta_0}^{\zeta} \left|f(t,\lambda_0)\right| \left|dt\right|\right|,$$

$$\leq M \int_{\zeta_0}^{\zeta} \left|dt\right| < MS \quad .$$

But by construction, $S \leq \frac{r}{M}$ so that

$$\left|\lambda_1 - \lambda_0\right| < r$$

and λ_1 is in D_1. We know that $f(\zeta,\lambda)$ is regular, continuous, and single valued in its second argument, provided the second argument is within D_1. But λ_1 is within D_1; consequently, $f(\zeta, \lambda_1)$ is a regular single-valued function of ζ within D and is continuous in the closure of the domain D. From this we may conclude that λ_2 is a regular single-valued function of ζ within D and a continuous function of ζ within and on the boundary of D.

Suppose now that λ_{n-1} is within D_1 and is a regular, single-valued function of ζ within D, and is continuous in the closure of D. Then

$$\left|\lambda_n - \lambda_0\right| \leq \left|\int_{\zeta_0}^{\zeta} \left|f(t,\lambda_{n-1})\right| \left|dt\right|\right|,$$

$$\leq MS < r,$$

so that λ_n also is within D_1. Further,

$$\lambda_n = \lambda_0 + \int_{\zeta_0}^{\zeta} f(t,\lambda_{n-1})\, dt \ ,$$

so that λ_n is also regular and single valued within D and continuous in the closure of D. Thus the induction is established.

We now show that these λ_n converge uniformly and absolutely to a limit function $\lambda(\zeta)$. In the remainder of this proof we will always agree to take the geodesic path when integrating to a point ζ. It is clear that

$$\left|\lambda_1 - \lambda_0\right| \leq \left|\int_{\zeta_0}^{\zeta} \left|f(t,\lambda_0)\right|\left|dt\right|\right|,$$

$$\leq M \int_{\zeta_0}^{\zeta} ds = Ms \quad ,$$

where s is the distance from ζ_0 to ζ along the geodesic path. Notice that s is a function of ζ and is not to be confused with S, which is a constant. If D is convex, then $s = \left|\zeta - \zeta_0\right|$. Now

$$\left|\lambda_2 - \lambda_1\right| \leq \int_{\zeta_0}^{\zeta} \left|f(t,\lambda_1) - f(t,\lambda_0)\right|\left|dt\right|,$$

and upon applying the Lipshitz condition we have

$$\left|\lambda_2 - \lambda_1\right| \leq K \int_{\zeta_0}^{\zeta} \left|\lambda_1 - \lambda_0\right|\left|dt\right|.$$

But we have just shown that $\left|\lambda_1 - \lambda_0\right| < Ms$, so

$$\left|\lambda_2 - \lambda_1\right| < MK \int_{\zeta_0}^{\zeta} s\, ds = MK \frac{s^2}{2!} \quad .$$

Suppose now that

$$\left|\lambda_{n-1} - \lambda_{n-2}\right| < MK^{n-2} \frac{s^{n-1}}{(n-1)!} \quad .$$

We have

$$\left|\lambda_n - \lambda_{n-1}\right| \leq \int_{\zeta_0}^{\zeta} \left| f(t,\lambda_{n-1}) - f(t,\lambda_{n-2})\right| \left|dt\right|,$$

$$\leq K \int_{\zeta_0}^{\zeta} \left|\lambda_{n-1} - \lambda_{n-2}\right| \left|dt\right|,$$

$$\leq MK^{n-1} \frac{1}{(n-1)!} \int_{\zeta_0}^{\zeta} s^{n-1}\, ds = MK^{n-1} \frac{s^n}{n!} .$$

We may now replace s by S so that we have a uniform bound for the absolute values of the differences $\lambda_n - \lambda_{n-1}$. Consequently the sum

$$\lambda_0 + \sum_{j=1}^{\infty} \left\{\lambda_j(\zeta) - \lambda_{j-1}(\zeta)\right\} ,$$

by the Weierstrass M-test, converges absolutely and uniformly in the closure of the domain D. Therefore, $\lambda_n \rightarrow \lambda(\zeta)$ and $\lambda(\zeta)$ is regular and single valued within D and continuous within and on the boundary of D, because a uniformly convergent sequence of continuous or regular functions converge to a continuous or regular function.

Consider the equation

$$\lambda_n(\zeta) = \lambda_0 + \int_{\zeta_0}^{\zeta} f(t, \lambda_{n-1}(t))dt .$$

If we take the limit as $n \rightarrow \infty$ of both sides we obtain

$$\lim_{n\to\infty} \lambda_n(\zeta) = \lambda_0 + \lim_{n\to\infty} \int_{\zeta_0}^{\zeta} f(t, \lambda_{n-1}(t))dt ,$$

or

(11) $$\lambda(\zeta) = \lambda_o + \int_{\zeta_o}^{\zeta} f(t,\lambda)dt .$$

This last step is legitimate by virtue of the uniform convergence of the functions λ_n.

If we consider the Lipshitz condition

$$\left|f(\zeta,\lambda_n) - f(\zeta,\lambda)\right| < K\left|\lambda - \lambda_n\right| < K\varepsilon_n ,$$

we see that the functions $f(\zeta,\lambda_o)$, $f(\zeta,\lambda_1)$, $f(\zeta,\lambda_2)$ ... form a uniformly convergent sequence of regular single-valued functions within D, continuous in the closure of D. Consequently $f(\zeta,\lambda)$, the limit function, is also regular and single valued within D and continuous in the closure of D. Under these conditions we may differentiate the indefinite integral (11) to obtain

$$\frac{d\lambda}{d\zeta} = f(\lambda,\zeta) .$$

Thus $\lambda(\zeta)$ satisfies the differential equation and from (11) it is clear that $\lambda(\zeta_o) = \lambda_o$.

Finally, we prove the uniqueness of λ. Let $\Lambda(\zeta)$ be a function which assumes the value λ_o at ζ_o and which satisfies equation (10), and hence equation (11). Suppose that Λ is regular, single valued, and continuous in a connected domain D* which includes those regions of D in which the inequality

$$\left|\Lambda - \lambda_o\right| < r$$

holds. At least a small region of D near ζ_o is included by reasons of

continuity. If ζ is now restricted to D*, we have

$$\Lambda - \lambda = \int_{\zeta_0}^{\zeta} [f(t,\Lambda) - f(t,\lambda)]dt \ ,$$

whence

$$\left|\Lambda - \lambda\right| \leq \int_{\zeta_0}^{\zeta} \left|f(t,\Lambda) - f(t,\lambda)\right| \left|dt\right|,$$

$$\leq K \int_{\zeta_0}^{\zeta} \left|\Lambda - \lambda\right| \left|dt\right|,$$

$$\leq Kr \int_{\zeta_0}^{\zeta} ds = Krs,$$

where s has the same meaning as before. But again

$$\left|\Lambda - \lambda\right| \leq \int_{\zeta_0}^{\zeta} \left|f(t,\Lambda) - f(t,\lambda)\right| \left|dt\right|,$$

$$\leq K \int_{\zeta_0}^{\zeta} \left|\Lambda - \lambda\right| \left|dt\right|.$$

We may now use the estimate above to obtain

$$\left|\Lambda - \lambda\right| \leq K^2 r \int_{\zeta_0}^{\zeta} s \ ds = K^2 r \frac{s^2}{2!} \ .$$

It is quite clear that the induction can be established so that we obtain, finally,

$$\left|\Lambda - \lambda\right| \leq K^n r \frac{s^n}{n!} \leq K^n r \frac{S^n}{n!} .$$

But the left-hand side is independent of n; consequently, $\Lambda = \lambda$ in D*. But if $\lambda = \Lambda$ in D*, then necessarily $\lambda = \Lambda$ in D. This completes the proof.

We now have the equipment to prove that a unique function $\lambda(\zeta)$ exists for any free-surface flow, and that the free surface is an analytic curve.*/ Our basic tool is the differential equation (7). We consider as given a complex potential $\zeta(z) = \varphi + i\psi$ which represents a free-surface flow, namely, which is analytic in some region of the z-plane bounded by an arc lying in the lower half-plane whose image is a segment of the real ζ-axis. Along this segment we assume that the equation

$$1/2 \left|\frac{d\zeta}{dz}\right|^2 + y = 0$$

holds. This is tantamount to saying that the pressure is constant along the arc. This exceptional arc will be called, and is, the free surface.

For the proof, it suffices to consider a small arc of free surface which is bounded in depth and which is bounded away from the real z-axis. That is to say, on this free-surface segment, $M \geq y \geq \varepsilon > 0$. It is then clear from Bernoulli's law that $z' = \dfrac{1}{\frac{d\zeta}{dz}}$ is also bounded and bounded away from zero. We will further assume that z' is continuous on the free surface. This means that the free surface must, in particular, have a continuously turning tangent. It is given that z' is analytic <u>within</u> some semi-neighborhood

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*/ The method used in proving this theorem has been presented in more generality by Lewy in A theory of terminals and the reflection laws of partial differential equations, Technical Report No. 4, Contract Nonr-225(11)(NR-041-086), Office of Naval Research, Applied Mathematics and Statistics Laboratory, Stanford University, California, Aug. 8, 1952.

of the free surface.

We first find the λ function that goes with this flow. Equation (7),

$$z'(z' + 2i\lambda') - \frac{1}{2\lambda} = 0, \tag{7}$$

may be considered as a differential equation for an unknown $\lambda(\zeta)$ with known z'. Solving for λ' we have

$$\lambda' = \frac{d\lambda}{d\zeta} = \frac{1}{2i}\left(\frac{1}{2z'\lambda} - z'\right). \tag{12}$$

For initial values, we will take a point on the free surface. Let $\lambda_o = -y_o$, where y_o is the imaginary part of a point z_o on the free surface which is the image of ζ_o. The region D_2 is some lower semi-neighborhood of the image of the free-surface arc under consideration where z' is regular, continuous, and single valued. It is clear that all the hypotheses about the function $f(\zeta, \lambda) = \frac{1}{2i}\left(\frac{1}{2z'\lambda} - z'\right)$ are satisfied if we take for D_1 any circle which excludes the origin of the λ-plane and for D_2 some lower semi-neighborhood of the real ζ-axis where, by reason of continuity, z' remains bounded away from zero and infinity as it is on the free-surface segment. Thus there exists a unique function $\lambda(\zeta)$ which satisfies equation (12) and attains the value $-y_o$ at ζ_o.

To see that this $\lambda(\zeta)$ becomes $-y$ all along the free-surface segment included in D and not just at the point z_o, we use the uniqueness of the solution. It suffices to show that $-y$ is a solution to equation (7) along the free-surface segment. Consider Bernoulli's law in the form

$$\frac{dz}{d\zeta}\,\frac{d\bar{z}}{d\bar{\zeta}} + \frac{1}{2y} = 0 .$$

Since we are only interested in staying on the free surface, $d\zeta = d\varphi$;

hence $d\bar{\zeta} = d\zeta$, and we may replace $\frac{d\bar{z}}{d\bar{\zeta}}$ by $\frac{d\bar{z}}{d\zeta}$. Further, $z-\bar{z} = 2iy$, and on the free surface we have assumed that the real and imaginary parts of z (now functions of the real variable φ alone) have continuous derivatives, so that $\bar{z}' = z' - 2iy'$ has meaning, where prime designates $\frac{d}{d\zeta} = \frac{d}{d\varphi}$. Thus Bernoulli's law takes the form

$$z' (z' - 2iy') + \frac{1}{2y} = 0 ,$$

which, upon replacing y by -y, is just equation (7). This verifies that -y is a solution of equation (7) and therefore that $\lambda(\zeta)$ and -y coincide on the free surface. Incidentally, this shows that $\lambda(\zeta)$ is independent of the starting point z_o.

As a function of ζ, λ is real on the real ζ-axis and hence can be continued into the upper half-plane by Schwarz's reflection principle. This defines $\lambda(\zeta)$ as an analytic function above and, in particular, on the real ζ-axis. We then return to equation (7) and consider it as a differential equation in an unknown $z(\zeta)$ and a known $\lambda(\zeta)$ above the free surface. As such it is solvable in the form of equation (8), which thus serves to determine z above the free surface. Because this new $z(\zeta)$ defined above the free surface agrees along the free surface with the original $z(\zeta)$ of the flow, it is, by definition, the analytic continuation of the original $z(\zeta)$. But $z(\zeta)$, being an analytic function, determines a hydrodynamical flow. Consequently, we have extended the flow across the free surface boundary. The continued function $\lambda(\zeta)$ is analytic on the free-surface segment, so by Morera's theorem and equation (8), $z(\zeta)$ is also analytic on the free surface. Since the free surface is the image of $\mathrm{Im}(\zeta) = 0$, it is represented in an analytic way by

$$z = z(\varphi) = x(\varphi) + iy(\varphi) ,$$

and is hence an analytic curve.

This proof breaks down completely if we allow the free surface to reach the line $y = 0$, that is, if we allow a stagnation point to occur on the surface, for then $\lambda = 0$ and the Lipshitz condition cannot be satisfied. As Stokes pointed out, the free surface may then have a cusp of angle $\frac{2\pi}{3}$ at this point. This can be seen as follows: Let the point in question be the origin in the ζ-, λ-, and z-planes. If the slope of the free surface near the stagnation point is bounded away from zero and infinity, then x and y will be of the same order there. Consequently, z will be of the same order as λ, that is,

$$z = \zeta^{\alpha} R_1(\zeta) \text{ and } \lambda = \zeta^{\alpha} R_2(\zeta) ,$$

where R_1 and R_2 are regular and bounded away from zero in the partial neighborhood of the origin within the region of flow. For the differential equation (7) to be satisfied, the order of the first term must be the same as the order of the last. If we substitute the values of z and λ given above, we obtain

$$\zeta^{\alpha-1} R_3(\zeta^{\alpha-1} R_3 - 2i\,\zeta^{\alpha-1} R_4) + \zeta^{-\alpha} R_5 = 0 \quad ,$$

where R_3, R_4, and R_5 have the same properties as R_1 and R_2. The order of the first term is thus $2\alpha - 2$ and the last term has the order $-\alpha$. Consequently, we require $2\alpha - 2 = -\alpha$, or $\alpha = 2/3$.

This is not a necessary condition for surface stagnation, for if we allow x and y to be of different orders, we may have no cusp at all. An example of this latter case will be given in Chapter V.

Chapter II

In this chapter we develop a representation for the flow under a free surface, the shape of which is a trochoid (curtate cycloid).*/ This particular example is of interest because it is related to an early attempt to find the exact equations for ocean waves. The original investigation was due to Gerstner [2] and was later duplicated independently by Rankine [7]. The main difficulty here was that these waves were rotational.

If one attempts to solve the exact equation of the classical ocean-wave problem, keeping only terms up to the second order, then one is led to the trochoidal surface. (Terms of the first order yield a sinusoidal surface.) These investigations are due mainly to Stokes [10] and Rayleigh [8].

The approach we will take here will be from the opposite point of view with respect to the classical theory. Instead of specifying that the bottom is flat and then searching for the surface, we will specify that the surface is a trochoid and find out what kind of bottom will cause this. It is known that the trochoid does not represent the true ocean wave form. (This chapter shows this.) So, we will answer the question, "What type of flow does produce the trochoid as a free surface in the exact irrotational theory?"

It is perhaps helpful to have a possible physical application in mind as we develop the theory. One could imagine an infinitely long washboard with a sheet of water flowing along it at a fixed velocity. The theory might apply to the flow of water in a river if it were sufficiently straight of

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*/ Fritz John discovered this flow, independently, using a method similar to that of this paper. His general method, however, treats unsteady flows and this example was a special case where the flow was made steady. John, Fritz "Two Dimensional Potential Flows with a Free Boundary," <u>Communications on Pure and Applied Mathematics</u>, Vol. VI (1953) pp. 497-503.

course, uniform in width and of proper speed, depth, and bottom shape.

However, the main practical value of this theory is its possible application to the study of ocean waves of great length and small height, such as those caused by underwater seismic disturbances. As is shown, the roughness of the bottom becomes infinitesimal with respect to the roughness of the surface if the ratio of wave height to wave length is sufficiently small. The flow of this chapter is useful more as an indication of how small this ratio must be in order to have the existing formulas apply than as a direct source of expressions useful in calculating the velocity of these waves, for velocity expressions have existed for some time (Stokes) which yield good results for waves of small wave height to wave length ratio. This chapter is also theoretically valuable, since it gives an estimation of how well the trochoidal curve approximates the profile of ocean waves.

The approach used here does not require the determination of a function $\lambda(\zeta)$, but is a direct procedure whereby we continue the flow away from the given free surface. Recall that in the existence proof we assumed that a flow was given together with its free surface. A consequence was that the free surface was analytic. Now it is known that an analytic function is completely determined if it is known along a curve. Thus actually the only part of a flow that needs to be given is the free surface itself, for then by means of this theory we are able to extend the flow under this free surface.

There are several methods applicable to this problem of continuation; however, they are all variations of the following idea. Since the free surface is analytic, it can be described by an analytic function $y = f(x)$. But $x = z - iy$, so that from $y = f(z - iy)$ we can solve for y to obtain y

as a function of z, $y = g(z)$. If we multiply equation (7) by $(\frac{d\zeta}{dz})^2$ we obtain

$$1 + 2i\frac{d\lambda}{dz} - \frac{1}{2\lambda}\left(\frac{d\zeta}{dz}\right)^2 = 0 ,$$

whence

$$\frac{d\zeta}{dz} = \sqrt{2\lambda + 2i\lambda\frac{d\lambda}{dz}} .$$

Replacing λ by $-g(z)$ we can solve this differential equation for $\zeta = \zeta(z)$, which is the required flow function. Incidentally, if we invert $\zeta = \zeta(z)$ to obtain $z = z(\zeta)$, then $\lambda(\zeta) = -g(z(\zeta))$. In general, to get from the free surface to the actual flow, one inversion and one quadrature are necessary.

The usual representation of a trochoid of fixed wave length 2π, wave height b, and of variable total depression c takes the form

$$x = -\theta + b\sin\theta ,$$

$$y = -c + b\cos\theta ,$$

where we require $c > b > 0$. The condition $c > 0$ follows from the requirement that the free surface lies below the x-axis, that is, $\lambda(\zeta) > 0$ for ζ real. The definition of a trochoid requires $c > b$. The limiting case $c = b = 1$ will be treated in the next chapter.

Thus, on the free surface we must have

$$z = -\theta + b\sin\theta + i(-c + b\cos\theta) ,$$

where, in particular,

$$\lambda = -(-c + b\cos\theta) .$$

We note first that

$$-\int (\lambda - c + 1)\, d\theta = \int (-1 + b \cos \theta)\, d\theta = -\theta + b \sin \theta = x,$$

so that if we are to have

$$z = -i\lambda + \int \sqrt{\frac{1}{2\lambda} - \lambda'^2}\; d\zeta$$

on the free surface, we must have

$$x = \int \sqrt{\frac{1}{2\lambda} - \lambda'^2}\; d\zeta = -\int (\lambda - c + 1)\, d\theta \,.$$

We may use this relation to determine what $\zeta = \zeta(\theta)$ must be in order to yield the trochoidal free surface. To do this, we first differentiate the equation with respect to θ, obtaining

$$\sqrt{\frac{1}{2\lambda} - \lambda'^2}\; \frac{d\zeta}{d\theta} = -\lambda + c - 1 \,,$$

or, squaring,

$$\left(\frac{1}{2\lambda} - \lambda'^2\right) \left(\frac{d\zeta}{d\theta}\right)^2 = \lambda^2 - 2(c-1)\lambda + (c-1)^2 \,.$$

Upon replacing λ' by $\frac{d\lambda}{d\zeta}$ on the left-hand side, we have

$$\frac{1}{2\lambda} \left(\frac{d\zeta}{d\theta}\right)^2 - \left(\frac{d\lambda}{d\theta}\right)^2 = \lambda^2 - 2(c-1)\lambda + (c-1)^2 \,,$$

or

$$\left(\frac{d\zeta}{d\theta}\right)^2 = 2\lambda \left\{\lambda^2 - 2(c-1)\lambda + (c-1)^2 + \left(\frac{d\lambda}{d\theta}\right)^2\right\} .$$

We now insert for λ the free-surface function $c - b \cos \theta$. After reduction (the third degree terms drop out by virtue of the Pythagorian indentity) we have

$$\left(\frac{d\zeta}{d\theta}\right)^2 = (2c - 2b \cos \theta)\,(b^2 + 1 - 2b \cos \theta).$$

In general, this yields a composite elliptic integral for ζ. However, if we choose $2c = b^2 + 1$, the right-hand side becomes a perfect square and ζ is given by the elementary integral

$$\zeta = \pm \int (b^2 + 1 - 2b \cos \theta)\, d\theta \,.$$

This special choice of c corresponds to a choice of the velocity of flow along our washboard. It is quite natural to expect some value of c to be favored, since we have already fixed the wave length. That this particular value of c is the "best" must remain a point of speculation. The numerical work of tracing stream lines in the case of $2c \neq b^2 + 1$ becomes unmanageable. Rough calculations gave a strong indication, however, that the value $c = \frac{b^2 + 1}{2}$ actually did yield a smoother bottom than other values.

In order to have $\mathrm{Im}(\zeta) < 0$ correspond to the flow region, we choose the negative sign for the integral and obtain, finally, the parametric representation for the flow,

$$\zeta = -(b^2 + 1)\,\theta + 2b \sin \theta \,,$$

$$z = -\theta + b \sin \theta + i \left(-\frac{b^2 + 1}{2} + b \cos \theta \right.\,.$$

The expression for the velocity is particularly simple for this flow. We have

$$\frac{d\zeta}{dz} = \frac{\frac{d\zeta}{d\theta}}{\frac{dz}{d\theta}} = \frac{2b \cos \theta - (b^2 + 1)}{-1 + b \cos \theta - i \sin \theta}$$

$$= \frac{b^2 - b\,(e^{i\theta} + e^{-i\theta}) + 1}{1 - be^{-i\theta}} = 1 - be^{i\theta} \,,$$

whence, since the velocity is the complex conjugate, we have

$$\text{velocity} = q_1 + i\, q_2 = 1 - be^{-i\bar{\theta}} .$$

Since a flow must be represented by a single-valued analytic function $\zeta = \zeta(z)$, we can expect trouble in the above flow at points where $\frac{dz}{d\theta}$ is zero. This occurs when

$$be^{-i\theta} = 1 ,$$

or, letting $\theta = \xi + i\eta$, when

$$e^{\eta} (\cos \xi - i \sin \xi) = \frac{1}{b} .$$

Since b is a positive real number, we require $\xi = 2n\pi$, and the trouble spots are hence the points

$$\theta = 2n\pi + i \log \frac{1}{b} .$$

It can be shown that the flow occurs over a multi-sheeted Riemann surface below these points. This is indeed unfortunate, for the velocity expression indicates that the bottom gets smoother as we go deeper. However, if we are to remain in the realm of physically possible flows, we must stay above the images of these points.

The actual picture of the flow can be made quite easily. We first pick a value for b, where 2b is the wave height. As an illustration we will use $2b = \frac{1}{\sqrt{2}}$. The equations become

$$\zeta = \frac{9}{8}\, \theta + \frac{1}{\sqrt{2}} \sin \theta ,$$

$$z = -\theta + \frac{1}{2\sqrt{2}} \sin \theta + i \left(-\frac{9}{16} + \frac{1}{2\sqrt{2}} \cos \theta \right) .$$

The free surface, $\mathrm{Im}(\zeta) = 0$, is obtained by taking $\mathrm{Im}(\theta) = 0$ or θ real. The parametric equations,

$$x = -\theta + \frac{1}{2\sqrt{2}} \sin\theta ,$$

$$y = -\frac{9}{16} + \frac{1}{2\sqrt{2}} \cos\theta ,$$

then describe the free surface. A set of values appears in Table 1. The corresponding curve is shown in Figure 1 on page 31. To find the stream lines below this free surface, we seek values of $\theta = \xi + i\eta$ which will keep the imaginary part of ζ a negative constant. We have

$$\zeta = -\frac{9}{8}(\xi + i\eta) + \frac{1}{\sqrt{2}} \sin(\xi + i\eta)$$

$$= -\frac{9}{8}\xi + \sin\xi \cosh\eta - i\left(\frac{9}{8}\eta - \frac{1}{\sqrt{2}} \cos\xi \sinh\eta\right) ;$$

consequently pairs (ξ, η) must be found such that

$$\frac{9}{8}\eta - \frac{1}{\sqrt{2}} \cos\xi \sinh\eta = k .$$

We may pick for k successive values only up to $k = .295$, for we may not go beyond $\theta = 2n\pi + i \log \frac{1}{r}$, whence

$$k \leq \frac{9}{8} \log 2\sqrt{2} - \frac{1}{\sqrt{2}} \sinh \log 2\sqrt{2} \cong .295 .$$

In Table 2, we compute several pairs (ξ, η) for $k = .25$ and the resulting "bottom" is shown in Figure 1.

In this particular example of the wave motion, the "bottom" is rather shallow and quite wavy, hence, not very useful in the study of ocean waves. However, by reducing the wave height, we get not only an increase in depth --

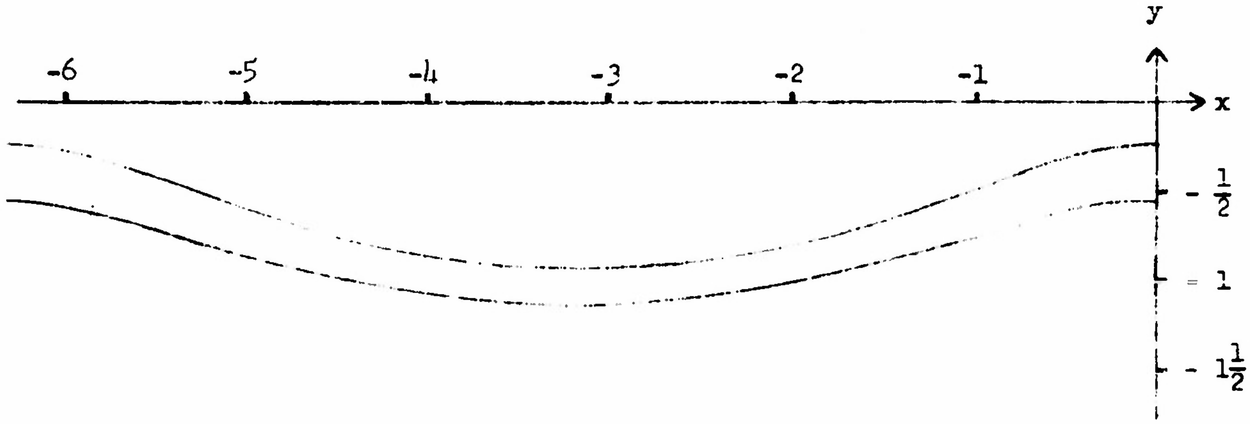

Figure 1

The trochoidal free surface and one stream line.
.05 units above the lowest possible stream line.

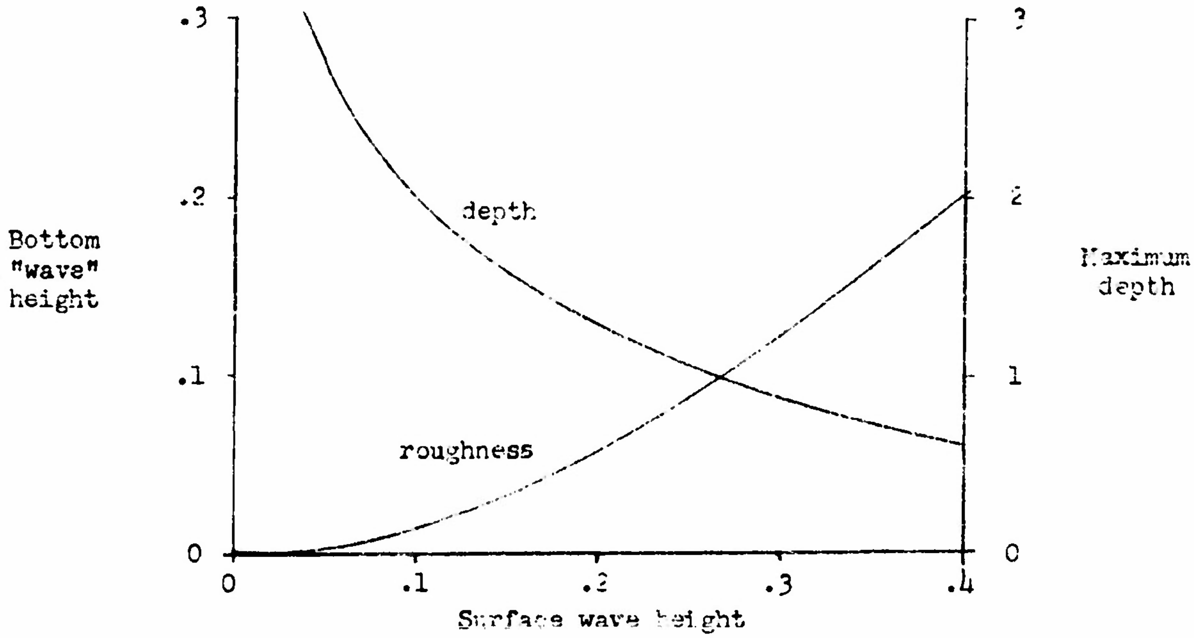

Figure 2

Bottom roughness and maximum depth as a function of wave height.

that is, the singularity in the flow moves down -- but percentagewise, we get a greater smoothing in the bottom. To compare the depth of singularity and smoothness of bottom for various values of wave height, b, we use the relations developed above with the special values $\xi = 0, \eta = \eta_1 = -\log b$ for the crest of the bottom curve, and $\xi = \pi, \eta = \eta_2$ (where η_2 is the value of η which yields the same imaginary value for $\mathcal{L}$ as $\eta_1 = -\log b$) for the trough of the bottom curve. The quantity $\eta^2 + \eta_2 + 1 - \eta_1$ is then the "wave height" of the bottom curve. Of course, 2b is the height of the surface wave and

$$(\eta_2 + \frac{b^2+1}{2} + be^{\eta_2}) - (\frac{b^2+1}{2} + b)$$

is the distance from trough of the surface wave to trough of bottom curve, this being what we take for the depth. It is, of course, not the usual definition of depth, but is, we think, a useful one for this type of flow. A table of values of these quantities, together with wave height to wave length ratio and percentage smoothing, is found in Table 3 for values of wave height ranging from .8 to .05, the wave length being fixed at 2π. A composite graph, Figure 2, displays some of this information. Of major interest is the velocity of propagation of these waves. Since in the treatment the coordinate system moves with the wave crests, we must look for the mean velocity of the water past the observer. Recall that

$$\vec{q} = \text{velocity} = q_1 + i\, q_2 = 1 - be^{-i\theta} .$$

Thus, the absolute value of the velocity along the free surface is

$$(q\,\bar{q})^{1/2} = \sqrt{1-2b \cos\theta + b^2} \quad \text{(for } \theta \text{ real)} .$$

The velocity at the crest is horizontal and of magnitude 1 - b, and the

velocity at the trough is also horizontal and of magnitude $1 + b$. The mean velocity along the free surface is

$$q_m = \frac{\int_c q_t \, ds}{\int_c ds} ,$$

where q_t is the tangential velocity, here the velocity itself, since the free surface is a boundary, and c is some length of arc representing a complete cycle. Since our wave is symmetric, we may use a half-wave length. Recall that on the free surface

$$x = \theta + b \sin \theta ,$$

$$y = -\frac{b^2 + 1}{2} + b \cos \theta ,$$

$$q = 1 - e^{-i\bar{\theta}} = 1 - e^{-i\theta} \text{ (for } \theta \text{ real)}$$

$$= 1 - b \cos \theta + ib \sin \theta = q_1 + iq_2 .$$

Hence,

$$q_m = \frac{\int_c q_1 \, dx + q_2 \, dy}{\int_c \sqrt{dx^2 + dy^2}} = \frac{-\int_0^{\pi} \left\{ (1 - b \cos \theta)^2 + b^2 \sin^2 \theta \right\} d\theta}{\int_0^{\pi} \left\{ (1 - b \cos \theta)^2 + b^2 \sin^2 \theta \right\}^{1/2} d\theta}$$

$$= \frac{-\int_0^{\pi} (1 + b^2 - 2b \cos \theta) \, d\theta}{\int_0^{\pi} (1 + b^2 - 2b \cos \theta)^{1/2} \, d\theta} .$$

The numerator can be integrated and yields the value $-\pi(1 + b^2)$. The denominator is an elliptic integral and, through the substitution $\theta = \pi - 2\gamma$, takes the form

$$2(1+b)\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{4b}{(1+b)^2}\sin^2\gamma}\,d\gamma = 2(1+b)\,E\left(\frac{2\sqrt{b}}{1+b}\right) .$$

Thus if we take the negative square root in this integral so that the signs will come out right, we find that the mean velocity of the fluid on the free surface is

(13) $$q_m = \frac{\pi}{2}\,\frac{1+b^2}{(1+b)^2}\left[E\left(\frac{2\sqrt{b}}{1+b}\right)\right]^{-1} .$$

Recall that b is the amplitude of the wave, 2b is the wave height. It should be pointed out that if we wish to interpret this flow as a progessive wave, we must make the wavy bottom move right along with the surface. Consequently, it is better to think of this flow as a river flowing over an undulating bottom. If the bottom of a surface-wave flow is flat or infinite in depth, then it is immaterial whether we consider the waves as fixed and the water in motion, or the water fixed and the waves in motion. Thus if we wish to compare the flow of this chapter with classical ocean waves, it is best to think of the ocean waves as progressing up a river which is flowing at the velocity of propagation of the waves. The bottom of the river must be flat and the river must be quite deep. It is of interest to compare velocities obtained from the various existing formulas with those obtained from equation (13). We consider these formulas for fixed wave length 2π and for $g = 1$. The quantity b is the amplitude of the wave, and the depth is assumed to be great with respect to the wave length in each case. From the linearized theory the velocity of propagation is independent of amplitude and is equal to one. Stokes' first approximation (yielding trochoidal waves) gave the velocity as $(1+b^2)^{1/2}$, and his second approximation, carrying terms up to the

fourth order, produced the relation $(1 + b^2 + \frac{5}{4} b^4)^{1/2}$ for the velocity. These latter expressions are compared in the accompanying table with values of q_m from equation (13) for various wave amplitudes.

b	.05	.1	.2	.3	.4	.5	.6	.7	.8
$(1+b^2)^{1/2}$	1.0013	1.005	1.020	1.044	1.077	1.12	1.17	1.22	1.28
$(1+b^2 + \frac{5}{4} b^4)^{1/2}$	1.0013	1.005	1.021	1.049	1.092	1.15	1.23	1.34	1.47
q_m	1.002	1.008	1.031	1.066	1.115	1.18	1.25	1.32	1.40

The correspondence in velocities is quite remarkable. For large values of b, however, the flow of this chapter cannot even be remotely compared with deep-water surface waves, since the bottom is very wavy and only a very short distance below the surface.

The results of this chapter would seem to indicate that either there is a remarkable similarity between flows over a washboard and deep-ocean waves, or the approximations used for ocean waves of relatively large amplitude are not as accurate as supposed.

Chapter III

Although the flow described in this chapter can be obtained directly from the example of Chapter II, we will develop it in all detail in order to illustrate a different method of attack. When one inserts a particular function $\lambda(\zeta)$ into the expression (8), he is often confronted by a highly complicated integral. In order to force this integral to be of a simple type, one could set the integrand equal to an elementary function. In this chapter we will set the integrand equal to unity.

We obtain, then, by equation (8), the system of equations

$$\frac{1}{2\lambda} - \lambda'^2 = 1 , \tag{14}$$

$$z = - i\lambda + \int d\zeta = - i\lambda + \zeta . \tag{15}$$

Here λ serves as a parameter to connect z and ζ. We notice that on the free surface, x is equal to ζ. One could set the integrand in equation (8) equal to any function of λ or ζ, provided the conditions stated in Chapter I are satisfied, namely, that the resulting expression obtained by solving the differential equation for λ is analytic in ζ, positive on some segment of the real ζ-axis, and satisfies the inequality $\frac{1}{2\lambda} - \lambda'^2 \geq 0$ on this segment.

If we solve the differential equation (14) for ζ, we obtain

$$\zeta = \int \frac{d\lambda}{\sqrt{\frac{1}{2\lambda} - 1}} . \tag{16}$$

Equation (15) can then be written

$$z = i\lambda + \int \frac{d\lambda}{\sqrt{\frac{1}{2\lambda} - 1}} . \tag{17}$$

It can be seen from equation (16) that if λ is restricted to the interval $0 \leq \lambda \leq 1/2$, ζ is real and varies between two values whose difference is $\pi/4$. Thus this portion of the real ζ-axis will correspond to the free surface. Equation (14) assures us that $\frac{1}{2\lambda} - \lambda'^2 \geq 0$ on the free-surface segment. Analyticity is obvious.

The integral (16) may be evaluated by means of the substitution $\lambda = 1/2 \cos^2 \theta$. We obtain

$$\zeta = -\int \cos^2\theta \, d\theta = -\frac{\theta}{2} - \frac{\sin 2\theta}{4},$$

and from (17),

$$z = -1/2 \cos^2\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4},$$

where these equations are parametric equations for the flow. With this change of parameter, the free surface is now represented by real values of θ, and the interior of the flow is obtained by allowing θ to range over certain complex domains which will be defined later. The equations may be further simplified by employing the identity $\cos^2\theta = 1/2\,(1 + \cos 2\theta)$ and substituting β for 2θ. We obtain

(18) $$\zeta = -1/4\,(\beta + \sin\beta);$$

(19) $$z = -1/4\,(\beta + \sin\beta) - \frac{i}{4}\,(1 + \cos\beta)$$

$$= -1/4\,[\beta + i(1 + e^{-i\beta})].$$

The shape of the free surface may be found by seperating z into its real and imaginary parts; this is possible on the free surface, since β is real there. We obtain

$$x = -1/4(\beta + \sin\beta) ,\qquad (20)$$
$$y = -1/4(1 + \cos\beta) .$$

These equations are easily recognized as the parametric equations for a cycloid.

We must now carefully determine just what part of the cycloid represents the free surface. Recall that λ is restricted to the range $0 \leq \lambda \leq 1/2$; consequently, since $\lambda = 1/2 \cos^2\theta$, this corresponds to some θ interval of length $\pi/2$ which we may take to be the interval $0 \leq \theta \leq \pi/2$, or in terms of β, $0 \leq \beta \leq \pi$. For this range of β, equations (20) will describe the half cycle of the cycloid between $x = 0$ and $x = -\pi/4$, y varying from $-1/2$ to 0. The continuation of this free stream line from $\beta = 0$ to $\beta = -\pi$, arises from taking the negative square root in equation (17). The segment of the real λ-axis from 0 to 1/2 is hence traversed twice. This corresponds to the segment of the ζ-axis lying between $-\pi/4$ and $\pi/4$. In order to investigate the flow further, we look for the image of the remaining part of the real ζ-axis. This will be the continuation, as a fixed boundary, of the free-surface stream line. It is no longer necessary to take λ into consideration, since the parameter β is all that is needed to connect z and ζ. We see from equation (18) that the continuation of the free surface as a fixed boundary corresponds to a curve in the complex β-plane along which $\mathrm{Im}(\zeta) = \mathrm{Im}\left\{-1/4(\beta + \sin\beta)\right\}$ will be equal to zero and along which $\mathrm{Re}\left\{-1/4(\beta + \sin\beta)\right\}$ varies between $-\infty$ and $-\pi/4$ and between $\pi/4$ and $+\infty$.

Since we will need the curves for which $\mathrm{Im}(\zeta) =$ constant, we will develop these curves now, and then we need only to set the constant equal to zero in order to obtain the equations for the continuation of the free surface. If, in equation (18), we set $\beta = \xi + i\eta$, we obtain

$$\mathrm{Im}(\zeta) = \mathrm{Im}\left\{-1/4\ (\xi + i\eta + \sin\xi \cosh\eta + i \cos\xi \sinh\eta)\right\}$$
$$= -\ 1/4\ (\eta + \cos\xi \sinh\eta)\ .$$

Setting our constant equal to $-\frac{k}{4}$ we obtain, for the stream lines, the relation

$$\eta + \cos\xi \sinh\eta = k\ ,$$

whence

$$\cos\xi = \frac{k-\eta}{\sinh\eta}\ ;\ \xi = \cos^{-1}\left(\frac{k-\eta}{\sinh\eta}\right)\ . \tag{21}$$

In order to find the image of these lines in the z-plane, we use equation (19) with $\beta = \xi + i\eta$ to obtain

$$z = -\ 1/4\ \left\{\xi + i\eta + i(1 + e^{\eta} e^{-i\xi})\right\}\ ,$$

or

$$-4z = \xi + e^{\eta} \sin\xi + i(1 + \eta + e^{\eta}\cos\xi)\ .$$

To write x and y as functions of a single parameter, we may use equation (21) together with the identity $\sin\xi = \pm\sqrt{1-\cos^2\xi}$, which yields

$$-4x = \cos^{-1}\left(\frac{k-\eta}{\sinh\eta}\right) \pm e^{\eta}\sqrt{1 - \frac{(k-\eta)^2}{\sinh^2\eta}}\ ,$$
$$-4y = 1 + \eta + e^{\eta}\ \frac{k-\eta}{\sinh\eta}\ .$$

For η very large, we see that x behaves asymptotically like e^{η} and that y is asymptotic to η. Consequently, for all values of k, including $k = 0$, the stream lines resemble, asymptotically, the logarithmic curve $y = \log x$. The flow is illustrated in Figure 3. Stagnation points occur at the ends of the free surface where the cycloid meets the fixed boundary. The angle here

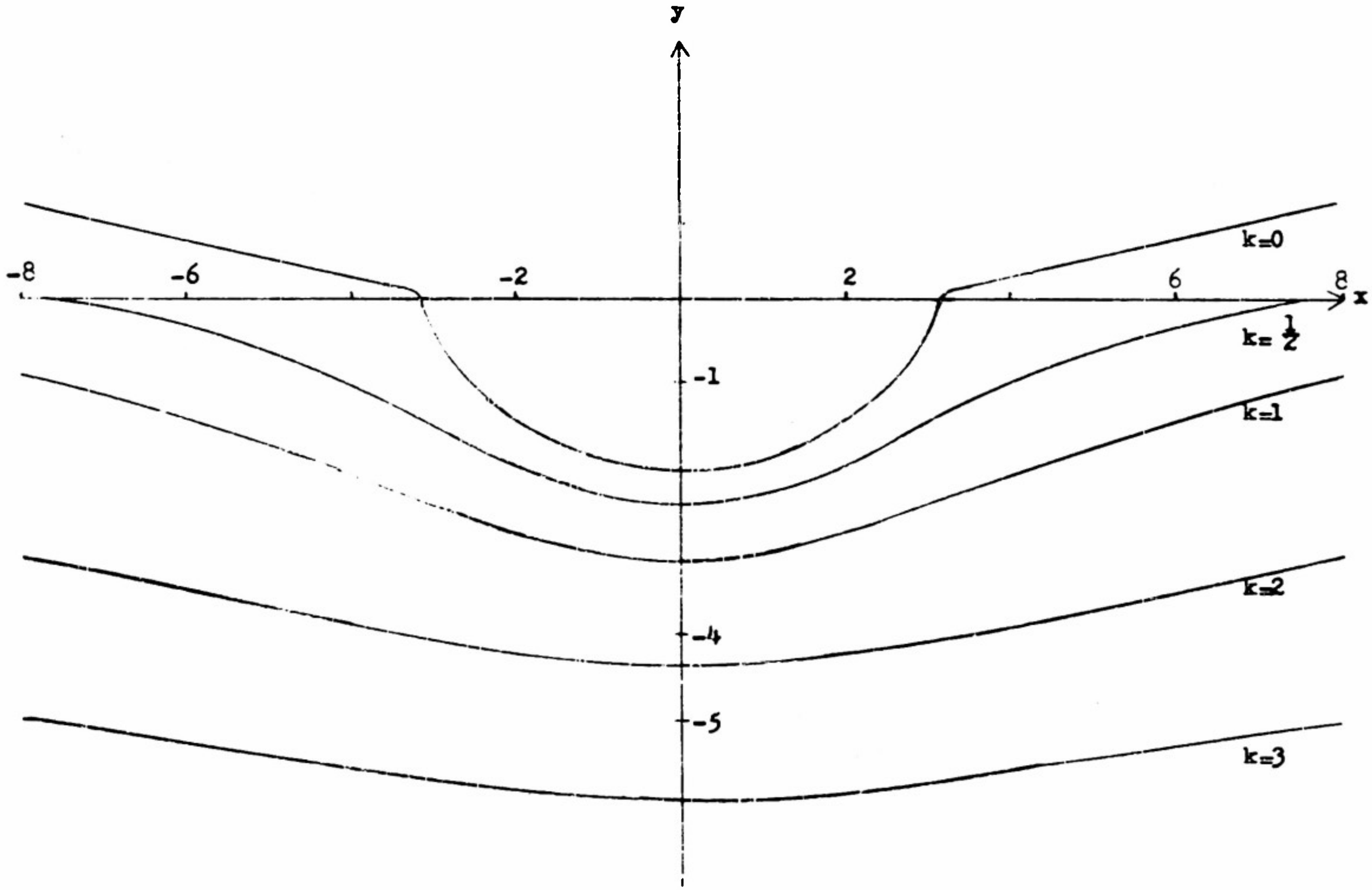

Figure 3

The cycloidal flow. The parts of the line $k = 0$ above the x-axis represent the "roof" and the part of the line $k = 0$ below the x-axis is the cycloidal free surface.

is $2\pi/3$. There are many branches to this mapping, but the only one involving a free surface and locally positive pressure is the one shown in Figure 3. It can best be interpreted as a flow of water under a "roof" shaped like the fixed boundary. In the break in the "roof." the water drops down in the form of a cycloid.

We find the velocity by taking the conjugate of

$$\frac{d\zeta}{dz} = \frac{\frac{d\zeta}{d\beta}}{\frac{dz}{d\beta}} = \frac{1 + \cos\beta}{1 + e^{-i\beta}} = 1/2(1 + e^{i\beta}) \quad ,$$

whence

$$2\vec{q} = 1 + e^{-\eta}\cos\xi - ie^{-\eta}\sin\xi \quad .$$

In order to investigate the other branches of the mapping, we expand z and ζ in power series about $\beta = \pi$. We have

$$-4\zeta = \beta + \sin\beta = \pi + \frac{(\beta-\pi)^3}{3!} - \frac{(\beta-\pi)^5}{5!} + \ldots ,$$

$$-4z = \beta + i(1 + e^{-i\beta}) = \pi + i\,\frac{(\beta-\pi)^2}{2!} + \frac{(\beta-\pi)^3}{3!} + \ldots .$$

If we set $|\beta - \pi| = \epsilon$, we may write */

$$-4z - \pi = i\frac{(\beta-\pi)^2}{2!} + O(\epsilon^3)$$

and

$$-4\zeta - \pi = \frac{(\beta-\pi)^3}{3!} + O(\epsilon^4) \quad .$$

If we let $\beta - \pi = re^{i\theta'}$, then

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*/ By $f = O(\alpha)$ we mean that $|f| \leq K|\alpha|$, K = constant, for all α sufficiently near to zero.

$$-4\zeta - \pi = \frac{r^3 e^{i3\theta'}}{3!} + O(\epsilon^4) .$$

Now the free surface and its continuations are the image of the lines $\mathrm{Im}(\zeta) = 0$; hence from

$$0 = \mathrm{Im}(\zeta) = \mathrm{Im}(-4\zeta - \pi) = \frac{r^3}{3!} \sin 3\theta' + O(\epsilon^4) ,$$

we see that necessarily $\sin 3\theta' = 0$, or $\theta' = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \ldots$.

Consequently, we have

$$-4z - \pi = i\frac{(\beta-\pi)^2}{2!} + O(\epsilon^3)$$

$$= i\,\frac{r^2}{2!}\, e^{i2\theta'} + O(\epsilon^3)$$

$$= \frac{r^2}{2!}\, e^{i(2\theta' + \pi/2)} + O(\epsilon^3) ; \qquad i = e^{i\pi/2} .$$

Thus in the neighborhood of $\beta = \pi$, that is, when $\epsilon \longrightarrow 0$, the argument of $-4z - \pi$, when z is tracing out a stream line, can assume the values $\pi/2$, $7\pi/6$, and $-\pi/6$. The branch corresponding to $\pi/2$ is the cycloidal free surface, and the branch corresponding to $-\pi/6$ is the "roof." The third branch, corresponding to $7\pi/6$, yields a flow which, for positive pressures, has no free surface at all. In order to see exactly what is happening near a branch point, it is sometimes convenient to follow a nearby stream line.

It is often helpful to trace out various stream lines in the parameter plane by composition of graphs. For example, in this flow we are interested in the lines for which

$$\mathrm{Im}(-4\zeta) = \eta + \sinh\,\eta \cos\xi = \text{constant} , \; \beta = \xi + i\eta .$$

One plots on the same graph the lines η = constant and the lines $\sinh\,\eta \cos\xi$ = constant for equally spaced increments. By joining the intersections where $\eta + \sinh\,\eta\cos\xi$ = constant, we construct the images of

the stream lines. Figure 4 illustrates this procedure.

For a flow to be physically realizable, we must investigate the pressures involved in the flow to make certain that they are positive. In order to do this we consider Bernoulli's law in the form

$$\text{constant} - \frac{p}{\rho} = 1/2\, q^2 + \mathrm{Im}(z) \quad .$$

If the pressure is to remain positive as we go deeper into the flow region, the quantity $1/2\, q^2 + \mathrm{Im}(z)$ must become large and negative there. In the case of this example, we have

$$1/2\, q^2 = 1/8\, (1 + 2e^{-\eta} \cos \xi + e^{-2\eta}) \ ,$$

and

$$\mathrm{Im}(z) = y = -1/4\, (1 + \eta + e^{\eta} \cos \xi) \ .$$

The velocity term decreases to 1/8 quite rapidly as we go into the flow, since η increases there (see Figure 4). As long as y is negative, the pressure is positive, but if y is large and positive, then the pressure will become negative. The latter is true when we are near the "roof" and far out on either side; consequently, this flow is not physically realizable in the large.

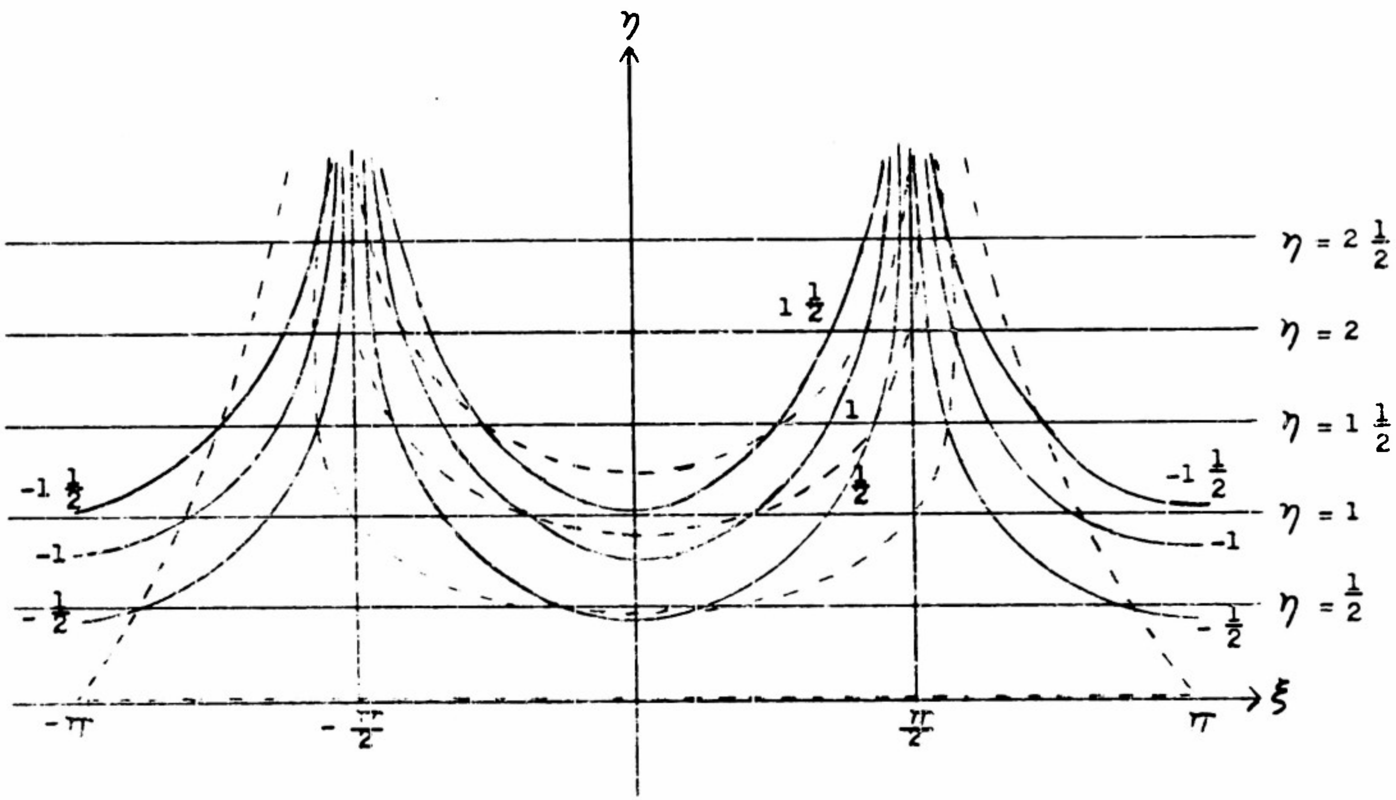

Figure 4

Construction of the stream lines in the β-plane. Dashed lines are the images of $\eta + \sinh\eta\cos\xi = k$ for $k = 0, 1, 2,$ and 3. The solid curved lines are graphs of $\sinh\eta\cos\xi = a$ for $a = 0$, $\pm\frac{1}{2}$, ± 1, and $\pm 1\frac{1}{2}$.

Chapter IV

As was mentioned in the introduction, one cannot hope to pick the one function $\lambda(\zeta)$ which will solve the ocean-wave problem unless the class of functions under consideration is restricted considerably. In this chapter, we will consider the influence that the problem of surface waves over a deep ocean has on the selection of the function $\lambda(\zeta)$. First we will show that under certain assumptions, $\lambda(\zeta)$ must be a function which has singularities. Then the exact location of these singularities will be determined and the behavior of $\lambda(\zeta)$ at the singularities will be found through the use of conformal mappings. We will gain enough knowledge about $\lambda(\zeta)$ to enable us to obtain a good approximation to the solution of the problem of waves over an infinitely deep ocean. An example is given illustrating the results of these considerations.

In order to establish our theorems about the behavior of $\lambda(\zeta)$, we must make an assumption concerning the physical flow. We assume that the waves are symmetric, in fact, that the whole flow is symmetric about the perpendicular line passed through the crest or trough of the wave. When one considers the fact that there are no external forces acting on the surface -- such as wind -- and that the water is frictionless for the purposes of this theory, this assumption of symmetry seems quite valid. A consequence of this symmetry is that within the frame work of this theory, all velocities are bounded and bounded away from zero. This does not mean that in the actual ocean zero velocities are not possible, for it must be kept in mind that our frame of reference is fixed with respect to the waves, not with respect to the water at great depth. Thus the fact that the velocity of the water is bounded away from zero means that in an actual ocean, the velocity of the water never

attains the velocity of propagation of the wave. This assumption appears, from observation, to be valid as long as the wave does not "break." The breaking of a wave apparently occurs when the velocity of the water at the crest of the wave exceeds the velocity of propagation of the wave.*/

The fact that the velocity is bounded has the same meaning with respect to either frame of reference.

As a result of our assumption of symmetry, we can show not only that the velocity is bounded and bounded away from zero in the entire flow region; we can determine the asymptotic development of $z = z(\zeta)$ near infinity as well. Suppose we consider the region in the z-plane bounded above by a half cycle of the free surface joining the trough to the crest of the wave, and on the sides by the two sem-infinite vertical lines dropped from the trough and the crest of the wave. These vertical lines are, as a consequence of our assumption of symmetry, orthogonal to every stream line, including the free surface.

Since the flow function $\zeta = \zeta(z)$ is an analytic function, it yields a conformal mapping; hence the region defined above in the z-plane maps into a region in the ζ-plane which is bounded on top by the image of the free surface, that is, by a segment of the real ζ-axis, and on the sides by lines that are everywhere orthogonal to the images of the stream lines. But then these lines must also be vertical straight lines dropped from the ends of the segment of the real ζ-axis that represents the half cycle of the free surface in the z-plane. We shall call the region in the z-plane, and

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*/Stokes has estimated that a wave will break when the ratio of wave height to wave length approaches 1:7.

its image in the ζ-plane, a "cell" of the flow.

We will locate the cell in the z-plane so that the vertical line under the trough of the wave coincides with the negative imaginary axis from $z = -ib$, $b > 0$, to $z = -i\infty$, and the vertical line under the crest of the wave runs from $z = d\pi - ia$, $0 < a < b$, to $z = d\pi - i\infty$. In the ζ-plane, the cell will be bounded by the segment of the real ζ-axis joining the origin to the point $\zeta = c\pi$, $c > 0$, and the two vertical lines dropped from these end points. The origin in the ζ-plane shall correspond to the point $z = -ib$ in the z-plane. Figure 5 on page 62 illustrates the cell as it appears in the z- and ζ-planes.

We now map the cell into two new regions by the mappings $w = e^{-iz/d}$ and $v = e^{-i\zeta/c}$. In the v-plane, the cell is mapped into the region bounded by the semi-circle of radius 1 lying above the real v-axis and the segment of the real v-axis joining $v = -1$ to $v = +1$. The image of the cell in the w-plane will be bounded below by a segment of the real w-axis joining $w = a'$ to $w = b'$, $-1 < a' < 0 < b' < 1$, and above by a smooth arc leaving the real w-axis at $w = a'$ vertically upwards and arriving at $w = b'$ vertically downwards, such that all points on the arc are less than unit distance from the origin. The image of the cell in the v- and w-planes is illustrated in Figure 6. The point at infinity in the z- and ζ-planes has as its image the origin in the v- and w-planes. The cell walls correspond to the segments of the real axes in the v- and w-planes.

If we consider the domains in the v- and w-planes as being connected by the mapping $\zeta = \zeta(z)$, which we assume as known, we see that w as a function of v is real on the real axis and consequently can be extended into the lower half-plane by the Schwarz reflection principle. By the same reasoning,

we can extend v into the lower half-plane. These extended domains are the unit circle in the v-plane, and a region bounded by a closed smooth arc in the w-plane containing the origin, but contained in the unit circle. Thus through the mapping $\zeta = \zeta(z)$ we can connect these extended regions in the v- and w-planes by an analytic function $w(v)$ which is schlicht within the image of the cell, and transforms the origin of the v-plane into the origin of the w-plane, and the positive direction of the real axis at the origin in the v-plane into the positive direction of the real axis at the origin in the w-plane. Since the mapping is schlicht, the derivative, $\frac{dw}{dz}$, is bounded and bounded away from zero within the image of the cell. Furthermore, the quantity $\frac{v}{w}$ is also bounded and bounded away from zero within the entire image of the cell, since at $w = v = 0$,

$$\lim_{w \to 0} \frac{v}{w} = \frac{dv}{dw}\Big]_{w=0} = \left(\frac{dw}{dv}\right)^{-1}\Big]_{w=0},$$

and we know that $\frac{dw}{dv}$ is bounded and bounded away from zero. But since

$$\frac{1}{\bar{q}} = \frac{dz}{d\zeta} = \frac{d}{c}\,\frac{v}{w}\,\frac{dw}{dv},$$

it is clear that provided c and d are finite and non-zero, the velocity, q, of the flow will also be bounded and bounded away from zero.

The mapping function $w(v)$ will have the form

$$w = a_1 v + a_2 v^2 + \ldots,$$

where a_1 is real and positive, and the series is convergent for $|v| < 1$. If we replace w and v by $e^{-iz/d}$ and $e^{-i\zeta/c}$, respectively, we obtain

$$e^{-iz/d} = a_1 e^{-i\zeta/c} + a_2 e^{-i2\zeta/c} + \dots = a_1 e^{-i\zeta/c}\left(1 + \frac{a_2}{a_1} e^{-i\zeta/c} + \dots\right).$$

Taking the logarithm of both sides, we find

$$-i\frac{z}{d} = \log\left(a_1 e^{-i\zeta/c}\right) + \log\left(1 + \frac{a_2}{a_1} e^{-i\zeta/c} + \dots\right),$$

whence

$$z = id \log a_1 + \frac{d}{c}\zeta + i\frac{da_2}{a_1} e^{-i\zeta/c} + \dots \tag{22}$$

Finally, we have

$$\frac{1}{q} = \frac{dz}{d\zeta} = \frac{d}{c} + \frac{d}{c}\frac{a_2}{a_1}\zeta e^{-i\zeta/c} + \dots,$$

which shows that the velocity at infinity is $\frac{c}{d}$.

Since velocity $= \overline{\left(\frac{d\zeta}{dz}\right)}$, we may conclude that in the equation

$$z'(z' + 2i\lambda') = \frac{1}{2\lambda}, \tag{7}$$

the quantity $z' = 1/\frac{d\zeta}{dz}$ is also bounded and bounded away from zero in the entire region of flow. We can now show that the function $\lambda(\zeta)$ must have a singularity in the finite part of the flow region of the ζ-plane. We will first show that $|\lambda| \leq M|\zeta| + N$ in this region for suitable constants M and N. If we solve equation (7) for λ', we obtain

$$\frac{d\lambda}{d\zeta} = \frac{1}{4iz'\lambda} - \frac{z'}{2i}. \tag{23}$$

From the form of this equation, and the bound on z' from above and below, we see that the right-hand side of the equation is bounded by some constant M as soon as λ is bounded away from zero. But then

$$\left|\lambda\right| \leq \int \left|\frac{1}{4iz'\lambda} - \frac{z'}{2i}\right| \left|d\zeta\right|$$
$$\leq M\left|\zeta\right| + N, \qquad N = \text{constant},$$

in the region of the cell excluding some circle about $\lambda = 0$. The inequality is obviously satisfied within the circle about $\lambda = 0$. Consequently, λ cannot increase any faster than ζ and is hence finite in the finite part of the flow region.

Now for a flow in an infinitely deep ocean, the entire lower half of the ζ-plane maps into the flow region, and the entire real ζ-axis represents the free surface. Therefore $\lambda(\zeta)$ is real along the entire real ζ-axis and is defined and bounded by $M\left|\zeta\right| + N$ in the lower half of the ζ-plane. As a consequence, according to Schwarz's reflection principle, we define $\lambda(\zeta)$ in the entire upper half-plane in a unique way by setting $\lambda = \overline{\lambda(\bar{\zeta})}$ there. Thus λ is defined and bounded by $M\left|\zeta\right| + N$ in the entire ζ-plane. If we were to assume that $\lambda(\zeta)$ were also <u>regular</u> in the entire ζ-plane, then by an extension of Liouville's theorem, λ must have the form

$$\lambda = A\zeta + B, \qquad A \text{ and } B \text{ constant.}$$

But this choice of the function $\lambda(\zeta)$ cannot yield a solution to the surface-wave problem, for it generates a free surface which is not even periodic. Thus we must conclude that λ cannot be regular within the region of flow. We will discuss the type and location of the singularities of $\lambda(\zeta)$ in more detail below.

If we again consider equation (23) and use the fact that z' is bounded, we see that if $\lambda = 0$, then necessarily $\lambda' = \infty$, that is, $\lambda(\zeta)$ is singular at the image of this point. Thus if we are to prevent any stagnation points from occurring in the flow, we must require that $\frac{1}{\lambda'} = \frac{d\zeta}{d\lambda} = 0$ at $\lambda = 0$.

We now turn our attention to the λ-plane and try to construct the image of the cell there. The part of the boundary that corresponds to the free surface is clearly a segment of the positive real axis, for we require $\lambda(\zeta)$ to be real and positive when ζ belongs to the free-surface segment. To find the remaining part of the cell, we return to the differential equation (7). As we move along one of the vertical boundaries of the cell, dz and dζ are both pure imaginary; consequently, along one of these lines $\frac{dz}{d\zeta}$ is real. This implies, in effect, that the velocity $\overline{\frac{d\zeta}{dz}}$ is real along the vertical wall of a cell. If we write I for known imaginary quantities and R_1, R_2,... for known real quantities, then on the sides of the cell equation (7) takes the form

$$R_1 \left(R_2 + 2i \frac{d\lambda}{I}\right) = \frac{1}{2\lambda} \quad ,$$

or, since $\frac{2i}{I}$ is real, we have

$$R_3 + R_4 \, d\lambda = \frac{1}{2\lambda} \quad .$$

Thus it seems that λ is real along the walls of the cell. That this is the case is a consequence of the uniqueness of the solution of equation (7). If one starts from either end of the free-surface segment and integrates down the side of a cell, one finds that in determining the successive approximations of Chapter I, page 14, all quantities, including the initial values, are real so that necessarily λ is also real along these vertical walls of the cell.

We can show that our cell boundary in the λ-plane is partially composed of the real λ-axis to the right of the free-surface segment, and the real λ-axis to the left of the free-surface segment up to the origin. At the origin of the λ-plane we must have a singularity, for as we found in the

discussion of the non-regularity of λ, it is necessary that $\frac{d\zeta}{d\lambda} = 0$ at $\lambda = 0$.

If we solve equation (7) for λ', we obtain

$$\lambda' = \frac{z'}{2i\lambda} \left(\frac{1}{2z'^2} - \lambda \right). \tag{24}$$

As we move downwards on either side of the cell, we have $d\zeta = id\Psi$, $d\Psi < 0$, so that, since z' and λ are positive near the free-surface segments, the sign of $d\lambda$ agrees with that of $\lambda - \frac{1}{2z'^2}$. But on these walls of the cells, the velocity, q, is real and equal to $\frac{d\zeta}{dz} = \frac{1}{z'}$; consequently, the sign of $d\lambda$ agrees with that of $\lambda - \frac{q^2}{2}$. As will be shown immediately, the velocity must decrease as we move down the cell wall under the trough of the wave, and increase as we move down the cell wall under the crest of the wave. At the trough of the wave, λ' and hence $d\lambda$ are zero by virtue of the form of Bernoulli's law on the free surface, namely, $\frac{1}{2}q^2 = -y$, and by the fact that $\lambda = -y$ on the free surface. Thus λ is stationary there. On the other hand, q decreases as we leave the trough and proceed down the wall of the cell. Consequently, $\lambda - \frac{q^2}{2}$ becomes positive as we leave the trough. Thus $d\lambda$ is positive and λ is increasing. But if q is decreasing and λ is increasing as we move down the wall of the cell, then $d\lambda$ will remain positive. As a result, λ must continue to increase until we reach a singularity. The right-hand side of equation (24) is regular along the wall of the cell below the trough of the wave, and hence there can be no singularity in λ along the positive real axis to the right of the free-surface segment, with the exception of the point at infinity. Thus the wall of the cell below the trough of the wave maps into the real axis in the λ-plane lying to the right of the point $\lambda = b$.

Below the crest of the wave, just the reverse holds. Here q is increasing so that the quantity $\lambda - \frac{q^2}{2}$ will change from zero to a negative quantity as

we leave the crest of the wave and move downwards on the wall of the cell. Thus $d\lambda$ is negative and λ is decreasing. Again, with λ decreasing and q increasing, $d\lambda$ remains negative and consequently λ will continue to decrease, at a rate bounded away from zero, until we encounter a singularity in equation (24). This singularity occurs when $\lambda = 0$. Thus the segment of the real axis from $\lambda = a$ to $\lambda = 0$ corresponds to part of the cell wall below the image of the crest of the wave. This proves that $\lambda(\zeta)$ has a singularity corresponding to $\lambda = 0$ which occurs directly below the image of the crest of the wave in the ζ-plane. Since we know that necessarily $\frac{d\zeta}{d\lambda} = 0$ at $\lambda = 0$, the image of the cell wall in the λ-plane must turn upwards at this point. We are unable to find the exact location of the cell wall beyond the point $\lambda = 0$, however. If we could find it, the problem of ocean waves would be virtually solved.

We will investigate this image further after we have shown that q increases as we move down the cell wall below the crest of the wave, and decreases as we move down the cell wall below the trough of the wave. To show this, we must assume that the wave rises continuously from trough to crest. This is surely the case for a simple wave. If we consider the vertical component of velocity, φ_y, we see that it must be positive on the free-surface segment and, as a consequence of our assumption of symmetry, it is zero along the cell walls and at infinity. But φ_y is a harmonic function, and consequently it must assume its minimum on the boundary. Thus $\varphi_y \geq 0$ within the cell. Furthermore, if a harmonic function assumes its minimum within a region, then it is necessarily constant. Consequently, φ_y is actually positive within the cell.

If we move infinitesimally from the cell wall below the trough of the wave into the interior of the cell region, then we are moving in the direction

of increasing x and increasing φ_y. Consequently, $\varphi_{yx} > 0$ on this cell wall. Conversely, if on the cell wall below the crest of the wave, we move into the interior of the cell, we are moving in the direction of decreasing x and increasing φ_y. Consequently, $\varphi_{yx} < 0$ on the cell wall below the crest of the wave. But $\varphi_{yx} = \varphi_{xy}$, so that $\varphi_{xy} > 0$ on the cell wall below the trough and $\varphi_{xy} < 0$ on the cell wall below the crest. This is just our assertion that the velocity, which is equal to φ_x on the cell walls, is decreasing as we move down the cell wall below the trough, and increasing as we move down the cell wall below the crest of the wave, since when we are moving downwards, we are moving in the direction of decreasing y.

Although we cannot find the exact image of the cell wall beyond $\lambda = 0$ in the λ-plane, we can assert that it must lie in the first quadrant, and must be an analytic arc. We will show that, in general, $\lambda(\zeta)$ is schlicht and possesses no singularities within or on the boundary of the cell besides the one on the cell wall below $\zeta = c\pi$ and the one at $\zeta = \infty$ which have already been mentioned. We will consider the mapping from the z-plane to the λ-plane. Since the mapping from the z-plane to the ζ-plane is regular and schlicht in the entire region of flow, what we assert for $\lambda = \lambda(z)$ is equally true for $\lambda = \lambda(z(\zeta)) = \lambda(\zeta)$.

If we multiply equation (7) by the quantity $\lambda(\frac{d\zeta}{dz})^2$, we obtain

$$\lambda + 2i\lambda\frac{d\lambda}{dz} = \frac{1}{2}\left(\frac{d\zeta}{dz}\right)^2 . \tag{25}$$

On the right-hand side of this equation, we replace $\frac{d\zeta}{dz}$ by $\varphi_x - i\varphi_y$. On the left-hand side, we set $\lambda = \lambda_1 + i\lambda_2$ and $dz = dx + idy$. We will consider the behavior of the imaginary part of equation (25) as we move vertically downwards in the z-plane, that is, for $dz = idy$, $dy < 0$. Under

this restriction, the imaginary part of the equation is

$$\lambda_2 + 2\frac{\partial}{\partial y}(\lambda_1\lambda_2) = -\varphi_x\varphi_y . \tag{26}$$

We have already seen that φ_y is positive in the interior of the cell and on the free-surface segment. We know from symmetry that φ_y is zero on the walls of the cell; consequently, since we have shown that the velocity is bounded and bounded away from zero, φ_x must have the same sign on each wall of the cell as it has at infinity. We have constructed our frame of reference so that $\varphi_x > 0$ at infinity, and hence φ_x is positive all along each wall of the cell. If we assume that the free surface does not have a vertical tangent, then φ_x is also positive along the free-surface segment. Consequently, φ_x is positive on the boundary of the cell, and hence, by the minimum principle for harmonic functions, φ_x is positive throughout the interior of the cell. Therefore the product $\varphi_x\ \varphi_y$ is positive in the interior of the cell and on the free-surface segment, and zero along the vertical walls of the cell.

Suppose now that we start from a point on the free surface in the z-plane and move downwards. We wish to show that λ_1 and λ_2 remain positive on any such vertical line dropped from the free surface other than the walls of the cell. From our construction of the image of the cell in the λ-plane, we know that λ_2 is zero on the free surface and that the region of flow lies above the free-surface segment in the λ-plane. If we write equation (26) in the form

$$\frac{\partial}{\partial y}(\lambda_1\lambda_2) = -\frac{1}{2}(\varphi_x\varphi_y + \lambda_2) , \tag{27}$$

we see that, since $\varphi_x\ \varphi_y$ is positive throughout the region of flow, and

since λ_1 is positive on the free-surface segment and λ_2 becomes positive as we leave the free-surface segment and enter the interior of the cell, the quantity $\frac{\partial}{\partial y}$ $(\lambda_1 \lambda_2)$ is negative as we start to move down a vertical line below the free surface in the z-plane. Because we are moving in the direction of decreasing y, this means that the product $\lambda_1 \lambda_2$ must be increasing. Now $\lambda_1 > 0$ on the free surface, $\lambda_2 = 0$ on the free surface; consequently, $\lambda_1 \lambda_2$ becomes positive as we leave the free surface. As long as λ_2 is positive, equation (27) shows that $\lambda_1 \lambda_2$ will continue to increase. Consequently, we can assert that λ_2 cannot be equal to zero in the interior of the cell, for if it were, then the quantity $\lambda_1 \lambda_2$ would have to increase to zero through positive values, an impossibility. With λ_2 positive, $\lambda_1 \lambda_2$ is constantly increasing so that we may conclude that λ_1 is also positive throughout the interior of the cell. Thus the image of the interior of the cell lies in the first quadrant, and λ is non-zero in the interior of the cell. From the form of the differential equation(23), we see that λ can have a singularity only at points where $\lambda = 0$ or at infinity. We have seen above that λ cannot be zero in the interior of the cell; consequently, if λ is also finite there, a fact that we have already established on page 50, then λ must be regular in the interior of the cell.

We have already seen that λ cannot be zero on the wall of the cell below the trough of the wave. We also found that λ <u>must</u> be zero at a point on the wall of the cell below the crest of the wave. We will now show that λ has only this one zero on this cell wall. As was pointed out earlier, the quantity $\varphi_x \varphi_y$ is zero all along the walls of the cell. Thus if we start from the image of $\lambda = 0$ on the wall below the crest of the wave and move downwards in the z-plane, equation (27) assumes the form

$$\frac{\partial}{\partial y}(\lambda_1 \lambda_2) = -\frac{1}{2} \lambda_2 .$$

We have already established the fact that λ_1 and λ_2 are positive within the cell. Consequently, on the boundary of the cell $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. As we leave the image of $\lambda = 0$ in the z-plane and proceed downward, λ_2 must assume positive values, for otherwise the cell boundary would double back on itself, and since the interior of the cell must lie to the right of the boundary, we would contradict the fact that λ_1 and λ_2 are positive in the interior of the cell. But if λ_2 becomes positive, then as seen before, the quantity $\lambda_1 \lambda_2$ must be increasing. Consequently, λ_1 cannot remain zero as we move away from the image of $\lambda = 0$ in the z-plane, but must also become positive. As long as λ_2 is positive, $\lambda_1 \lambda_2$ is increasing. Thus λ cannot be zero again along the wall of the cell below the image of $\lambda = 0$. Even more is true, namely, if the quantity $\lambda_1 \lambda_2$ is constantly increasing, then the image of the cell wall beyond $\lambda = 0$ must intersect every hyperbola $\lambda_1 \lambda_2 = k$ from left to right.

Since $\lambda_1 \lambda_2$ is the imaginary part of λ^2, we see that the image of the cell wall in the λ^2-plane is always rising. Consequently, in the mapping from z to λ^2 or from ζ to λ^2, the points on the boundary of the cell map into each other in a one-to-one continuous manner. But if this is the case, the mapping is schlicht in the interior of the cell as well.*/ If the mappings from z to λ^2 and ζ to λ^2 are schlicht, then so are the mappings from z to λ and ζ to λ.

That λ is bounded in the finite part of the ζ-plane, and hence can

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*/ See, for example, Titchmarsh, <u>The Theory of Functions</u>, Oxford, 1932, p. 201.

have no poles in the finite part of the ζ-plane, is an immediate consequence of equation (23), as was pointed out earlier. This completes the proof that the function $\lambda(\zeta)$ can have only one singularity in the finite part of the cell in the ζ-plane, and that it must have this singularity below the image of the crest of the wave. Further, we have seen that $\lambda(\zeta)$ is a schlicht function in the cell.

We have already shown that $|\lambda| \leq M|\zeta| + N$ for large values of λ. We will now show that λ becomes infinite like ζ at infinity. If we multiply equation (24) by $2i\lambda$, we obtain

$$2i\lambda\lambda' = i\frac{d\lambda^2}{d\zeta} = z'\left(\frac{1}{2z'^2} - \lambda\right).$$

At the trough of the wave, we have seen that λ is stationary and that the velocity decreases as we move down the wall of the cell below the trough. As we move down this cell wall, $d\zeta = id\Psi$, $d\Psi < 0$, so that λ^2 is increasing if $\frac{1}{2z'^2} = \frac{q^2}{2} < \lambda$ there. As we have seen, this is indeed the case. If λ^2 is increasing, so is λ, for λ is real and positive on the cell wall below the trough of the wave. With λ increasing and $\frac{q^2}{2}$ decreasing as we move down the cell wall, we are assured of finding some positive constant M_1 such that

$$i\frac{d\lambda^2}{d\zeta} = z'\left(\frac{1}{2z'^2} - \lambda\right) \leq M_1 \text{ for } \lambda \geq b^* > b,$$

or, since $d\zeta = id\Psi$, $d\Psi < 0$,

$$\frac{d\lambda^2}{|d\Psi|} \geq M_1 \text{ for } \lambda > b^*.$$

Consequently,

$$\lambda^2 \geq M_1 |\Psi| + M_2 \,, \quad M_2 = \text{constant.}$$

Since Ψ becomes large and negative as we move down the cell wall, λ becomes large and positive with at least the order of $\zeta^{1/2}$. If we write equation (7) in the form

$$\frac{d}{d\zeta}\,(z + 2i\,\lambda) = \frac{1}{2\,\lambda\,z'} \,, \tag{28}$$

we see that

$$z + 2i\lambda = \int 0\,(|\zeta|^{-1/2})\,d\zeta = 0\,(|\zeta|^{1/2}) \,.$$

We have already seen by equation (22) that for large values of ζ,

$$z = \frac{d}{c}\,\zeta + 0\,(1) \,;$$

consequently, we may conclude that

$$2i\,\lambda = -\frac{d}{c}\,\zeta + 0\,(|\zeta|^{1/2}) \,.$$

By reinserting this estimate into equation (28), we have that

$$\frac{d}{d\zeta}\,(z + 2i\,\lambda) = 0\,(|\zeta|^{-1}) \,,$$

from which we may conclude that

$$z + 2i\,\lambda = 0\,(\log|\zeta|) \,.$$

Thus near infinity

$$\lambda = \frac{i}{2}\,z + 0\,(\log|\zeta|)$$

$$= \frac{i}{2}\,\frac{d}{c}\,\zeta + 0\,(\log|\zeta|) \,.$$

Thus we find that not only does λ become infinite at infinity, we see even the behavior of the mapping near infinity. The width of the cell in the

λ-plane is just half the width of the cell in the z-plane as we move off to infinity. Since the width of the cell in the z-plane is one-half wavelength, the width of the cell at infinity in the λ-plane is $\frac{1}{4}$ wavelength.

Finally, we shall establish the behavior of $\lambda(\zeta)$ at the image of $\lambda = 0$. We find that we must have a square root singularity at the origin in the λ-plane, that is, the image of the cell wall turns vertically upwards there. We start with the differential equation (23), but write the right-hand side as a single fraction. We obtain

$$\frac{d\lambda}{d\zeta} = \frac{2iz'^2\lambda - i}{4z'\lambda} .$$

We now invert this differential equation and consider it as a differential equation for ζ as a function of λ. We have

$$\frac{d\zeta}{d\lambda} = \frac{4z'\lambda}{2iz'^2\lambda - i} , \tag{29}$$

where z' is considered as a known function of the dependent variable ζ. This differential equation is regular at $\lambda = 0$ and consequently will have a regular solution there of the form

$$\zeta = \zeta_0 + a_1\lambda + a_2\lambda^2 + \dots .$$

From equation (29), we see that $\frac{d\zeta}{d\lambda} = 0$ at $\lambda = 0$, whence $a_1 = 0$. Also, $\frac{d\zeta}{d\lambda}$ has a simple zero, so $a_2 \neq 0$. Thus near $\lambda = 0$ we have

$$\zeta - \zeta_0 = a_2\lambda^2 + a_3\lambda^3 + \dots,$$

which proves that the singularity at $\lambda = 0$ is a branch point of the first order.

This concludes our investigation of the function $\lambda(\zeta)$ for the problem of ocean waves over an infinitely deep ocean. We have found that in the

λ-plane, the cell boundary is composed of the real λ-axis and a curve which leaves the origin of the λ-plane vertically upwards, turns immediately to the right and asymptotically parallels the positive real λ-axis at a distance above it equal to one-half the width of the cell in the z-plane. W h this information, we are able to construct a free-surface flow which, because of our lack of knowledge of the image of the cell wall beyond $\lambda = 0$, is not the solution to the problem of ocean waves, but which does yield a good approximation to the solution to this problem.

We are concerned only with the mapping connecting the ζ- and λ-plane, for this will determine our choice for the arbitrary function $\lambda(\zeta)$. We assume that the ends of the free-surface segment in the ζ- and λ-planes are, respectively, 0 and $c\pi$, and a and b. The cell boundary in these two planes is illustrated in Figure 7, page 62. The singularity at the origin of the λ-plane is of the order of $\sqrt{\zeta}$.

In order to determine the mapping function, it is more convenient to rotate the ζ-plane into a w-plane, 90° removed from the ζ-plane, by the mapping $w = i\zeta$. Further, it is more convenient to place the singularity corresponding to $\lambda = 0$ in the w-plane, so that we can map a polygon in the w-plane into the real axis of the λ-plane. Figure 8 then illustrates the figures to be joined. We must map the real axis of the λ-plane into a polygon in the w-plane that is composed of the real positive w-axis, the part of the imaginary axis between the origin and the point $ic\pi$, the line joining $ic\pi$ to a point $k + ic\pi$, $k > 0$, and a ray leaving the point $k + ic\pi$ in the negative real direction. We will require the vertices of the polygon at the origin and $ic\pi$ to correspond to two positive points, b and a, $b > a$, on the real λ-axis, respectively. The vertex at $k + ic\pi$ will correspond

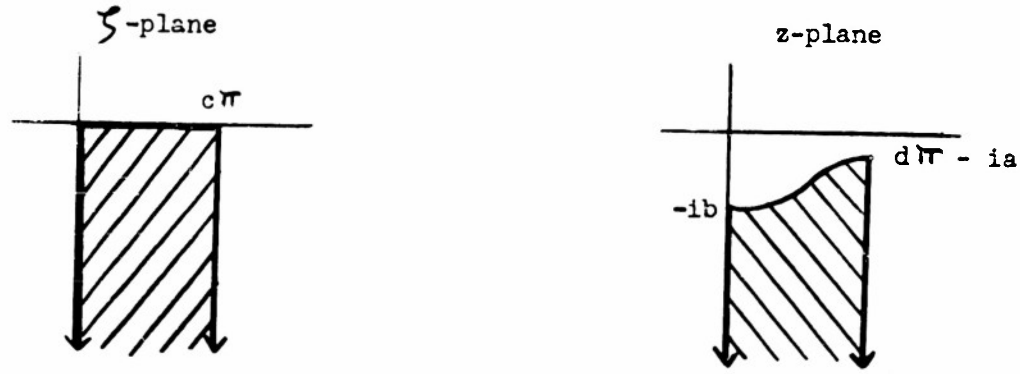


Figure 5

Figure 6

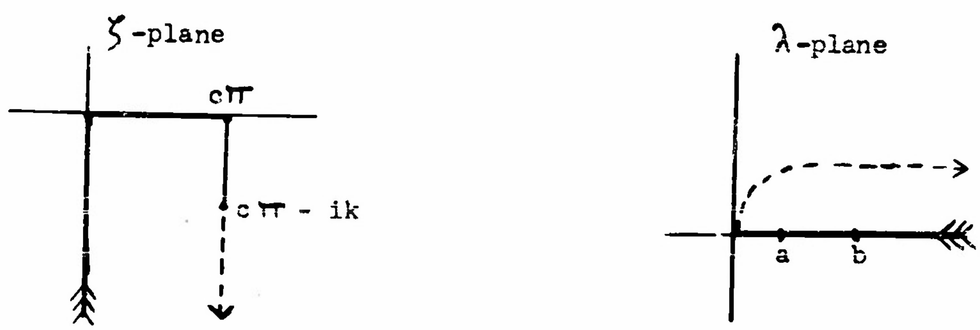


Figure 7

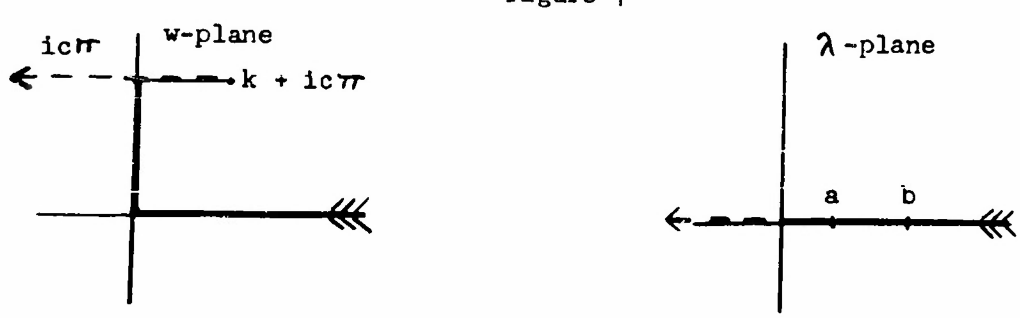


Figure 8

to the origin in the λ-plane. A value for k cannot be assigned -- its value is determined by a and b, and c.

Since we are mapping a polygonal domain onto the upper half-plane, we may employ the Schwarz-Christoffel transformation. The angles at the origin and at the point $ic\pi$ are both equal to $\pi/2$. The angle at $k + ic\pi$ is equal to $-\pi$. Thus the Schwarz-Christoffel transformation takes the form

$$w = A\int \lambda (\lambda - a)^{-1/2} (\lambda - b)^{-1/2}\, d\lambda + B,$$

where A and B are evaluated by the point correspondence $\lambda = b \rightarrow w = 0$; $\lambda = a \rightarrow w = ic\pi$. We integrate the transformation and obtain

$$w = A\left\{ \sqrt{(\lambda - a)(\lambda - b)} + (a + b) \log \left[\sqrt{\lambda - a} + \sqrt{\lambda - b} \right] \right\} + B .$$

The required point correspondence yields

$$A = \frac{2c}{a + b} , \quad B = - c \log (b - a) ,$$

whence, returning to ζ by means of the relation $\zeta = - iw$, we have

$$(30) \qquad \zeta = - \frac{2ic}{a + b} \sqrt{(\lambda - a)(\lambda - b)} - 2ic \log(\sqrt{\lambda - a} + \sqrt{\lambda - b}) + ic \log(b-a)$$

$$= - ic \left\{ \frac{2}{a + b} \sqrt{(\lambda - a)(\lambda - b)} + \log \frac{(\sqrt{\lambda - a} + \sqrt{\lambda - b})^2}{b - a} \right\} .$$

This last equation, or rather its inverse, is our choice for the arbitrary function $\lambda = \lambda(\zeta)$.

We must now check if this choice of the function $\lambda(\zeta)$ satisfies the necessary free-surface conditions. The mapping itself takes care of the requirement that λ is positive for ζ real. That $\frac{1}{2\lambda} \geq \lambda^{1/2}$ on the free surface-segment can be verified directly. We have

(31) $$\frac{d\zeta}{d\lambda} = -\frac{2ic}{a+b}\;\frac{\lambda}{\sqrt{(\lambda-a)(\lambda-b)}}\,,$$

and, consequently,

(32) $$\lambda'^2 = \left(\frac{d\lambda}{d\zeta}\right)^2 = \frac{-(a+b)^2(\lambda-a)(\lambda-b)}{4c^2\lambda^2}\,.$$

Thus the inequality $\frac{1}{2\lambda} \geq \lambda'^2$ becomes

$$\frac{1}{2\lambda} \geq -\frac{(a+b)^2(\lambda-a)(\lambda-b)}{4c^2\lambda^2}$$

or, since $0 < a \leq \lambda \leq b$ on the free surface, we have

$$c^2 \geq \frac{(a+b)^2(\lambda-a)(b-\lambda)}{2\lambda}\,.$$

The right-hand side attains its maximum for $\lambda = \sqrt{ab}$, where we must verify that

$$c^2 \geq \frac{(a+b)^2(\sqrt{ab}-a)(b-\sqrt{ab})}{2\sqrt{ab}} = \frac{[(a+b)(\sqrt{b}-\sqrt{a})]^2}{2}\,.$$

We will choose $a + b = 1$ and $c = 1$ for reasons to be given shortly. Thus with this restriction on a and b, the free-surface condition is satisfied.

To find z as a function of λ we must compute $\left(\frac{d\lambda}{d\zeta}\right)^2$ and $d\zeta$ as functions of λ. The first, equation (32), has already been found in connection with the free-surface inequality. From $d\zeta = \frac{d\zeta}{d\lambda}d\lambda$ and equation (31), we have

$$d\zeta = -\frac{2ci\lambda\, d\lambda}{(a+b)\sqrt{(\lambda-a)(\lambda-b)}}\,.$$

Consequently,

$$z = -i\lambda + \int\sqrt{\frac{1}{2\lambda}-\lambda'^2}\;d\zeta$$

$$= - i\lambda + \int \sqrt{\frac{1}{2\lambda} + \frac{(a+b)^2 (\lambda-a)(\lambda-b)}{4c^2 \lambda^2}} \left(- \frac{2ci\,\lambda\, d\lambda}{(a+b)\sqrt{(\lambda-a)(\lambda-b)}}\right)$$

$$(33) \qquad = - i\lambda - i\int \sqrt{\frac{\lambda^2 + [\frac{2c^2}{(a+b)^2} - (a+b)]\lambda + ab}{(\lambda-a)(\lambda-b)}}\, d\lambda \;.$$

From our form of Bernoulli's law and the fact that $\lambda = -y$ on the free surface, we see that the velocity at the trough of the wave (i.e., for $\lambda = b$) is $\sqrt{2b}$ and the velocity at the crest of the wave is $\sqrt{2a}$. If we compute the quantity $\overline{q} = \frac{d\zeta}{dz} = \frac{d\zeta}{d\lambda}\Big/\frac{dz}{d\lambda}$ and take its limit as $\lambda \longrightarrow \infty$ along the real axis, we obtain

$$\overline{q}_\infty = \frac{A}{2} = \frac{c}{a+b} \;.$$

Now it seems quite logical that if our flow is to represent a true physical flow, the velocity at infinity must be some intermediate value between the velocities at the crest and the trough of the wave. As it is not generally known what intermediate value should be taken, we will let the wave degenerate into a flat surface, and then equate the flow on the surface with the flow at infinity. Thus we set $a = b$ and obtain the relation

$$\sqrt{2b} = \frac{c}{2b} \;.$$

If we set $c = 1$, this is satisfied for $b = \frac{1}{2}$. Consequently, in the case $c = 1$, if there is no wave at all, the velocities at infinity and at the free surface are equal if they are both equal to one. It seems logical, then, to require the velocity at infinity to be equal to one when there are waves on the surface. Consequently, we set $a + b = 1$. With this restriction,

the velocity at infinity becomes the mean square of the velocities at the crest and trough of the wave.

We are now ready to compile our formulas and compute the actual flow. We note that since the flow region is above the real λ-axis, the arguments of $\lambda - b$ and $\lambda - a$ change from zero to π as we pass over the points b and a, respectively, from right to left. Consequently, $\sqrt{\lambda - b}$ becomes $i\sqrt{b - \lambda}$ as we descend past b, and $\sqrt{\lambda - a}$ becomes $i\sqrt{a - \lambda}$ as we descend past a. Therefore, the various branches of the mapping in the z-plane are

$$z = -i\lambda - i\int_b^{\lambda} \sqrt{\frac{(x+a)(x+b)}{(x-a)(x-b)}}\,dx \ , \quad \text{for } \lambda \geq b \ ,$$

$$z = -i\lambda + \int_{\lambda}^{b} \sqrt{\frac{(x+a)(x+b)}{(x-a)(b-x)}}\,dx \ , \quad \text{for } a \leq \lambda \leq b \ ,$$

and

$$z = \int_a^b \sqrt{\frac{(x+a)(x+b)}{(x-a)(b-x)}}\,dx - i\lambda - i\int_{\lambda}^{a} \sqrt{\frac{(x+a)(x+b)}{(a-x)(b-x)}}\,dx \ ,$$

$$\text{for } 0 \leq \lambda \leq a \ .$$

By means of these formulas, we may compute z for values of λ along the real λ-axis. This corresponds to computing z along the walls of the cell, up to the singularity at $\lambda = 0$. The integrals can be reduced to the sum of an elementary integral, an incomplete elliptic integral of the first kind, and an incomplete elliptic integral of the third kind. Unfortunately, this latter integral cannot be found in tabulated form. It was necessary to compute these integrals by machine in order to learn anything about the flow. The data that were computed yielded the shape of the free surface and the points of intersection of the stream lines with perpendiculars dropped from the wave crest and wave trough.

To find the points of intersection of the stream lines with the vertical cell walls in the z-plane, it is necessary to compute pairs of values of λ -- one less than a and the other greater than b -- which yield the same value for the imaginary part of ζ. The two branches of the function $\zeta = \zeta(\lambda)$ (equation (30)) to be used in this calculation are

$$\zeta = -i\left\{\frac{2}{a+b}\sqrt{(\lambda-a)(\lambda-b)} + \log\frac{(\sqrt{\lambda-b}+\sqrt{\lambda-a})^2}{b-a}\right., \text{ for } \lambda \geq b,$$

and

$$\zeta = \pi - i\left\{\frac{-2}{a+b}\sqrt{(a-\lambda)(b-\lambda)} + \log\frac{(\sqrt{b-\lambda}+\sqrt{a-\lambda})^2}{b-a}\right.,$$

for $0 \leq \lambda \leq a$.

The point in the ζ-plane that corresponds to $\lambda = 0$ is hence

$$\zeta = \pi - i\left\{\frac{-2\sqrt{ab}}{a+b} + \log\frac{(\sqrt{b}+\sqrt{a})^2}{b-a}\right\}.$$

As the wave height diminishes, that is, as $b - a \rightarrow 0$, this singularity moves off to infinity like $\log\frac{1}{b-a}$. As in the case of the trochoidal flow of Chapter II, the flow has, in general, no physical reality below the stream line that passes through the image of $\lambda = 0$ in the z-plane. Since $\frac{dz}{d\lambda} = 0$ for $\lambda = 0$, we cannot invert $z = z(\lambda)$ into a function $\lambda = \lambda(z)$ in the neighborhood of $\lambda = 0$. Thus we cannot find the complex potential $\zeta = \zeta(\lambda(z))$ in this region.

It is interesting to note that the inversion of $z(\lambda)$ will always be impossible at $\lambda = 0$ if we are to require that all velocities are bounded and bounded away from zero. Recall that at $\lambda = 0$, we require $\frac{d\zeta}{d\lambda} = 0$ if the velocity is to be bounded away from zero. In order to be able to invert $z = z(\lambda)$ at $\lambda = 0$, we must have $\frac{dz}{d\lambda} \neq 0$ at $\lambda = 0$. But then

$$\bar{q} = \frac{\frac{d\zeta}{d\lambda}}{\frac{dz}{d\lambda}} \to 0$$

which contradicts the fact that the velocity is bounded away from zero. In the example of this chapter, the velocity at $\lambda = 0$ is equal to $2\sqrt{ab}$.

For the actual computation, a wave height of .2 was chosen, so that a = .4 and b = .6. The results of the numerical integration can be found in Tables 5 and 6, and the resulting flow is illustrated in Figure 9 on page 69. This wave is a slightly better approximation for shallow water waves than was the trochoidal wave of Chapter II. In the case of the trochoidal wave, the stream line through the singularity at $\lambda = 0$ had a "wave height" of .0585 units when the surface wave had a wave height of .2 units. The corresponding figure for the example of this chapter is .044 units. The wave length and depth are approximately the same for the two flows.

It is interesting to note that the wave heights (Δy of Table 5) of the individual stream lines do not decrease monotonically as we move down away from the free surface, but exhibit a small oscillatory behavior. This might indicate that the velocity of the wave should be chosen slightly differently. On the other hand, it might be that if one could accentuate this oscillatory action, one could obtain a flat bottom at the minimum of the first oscillation, thus obtaining the exact form of waves over shallow water.

If in equation (33) we were to choose a value of c that would make the numerator a perfect square ($c^2 = \frac{1}{2}[\sqrt{a} + \sqrt{b}]\,[a + b]$), then the resulting flow is the trochoidal flow of Chapter II. This trochoidal flow has the property that the velocity at infinity is independent of the wave height, if we hold the wavelength constant.

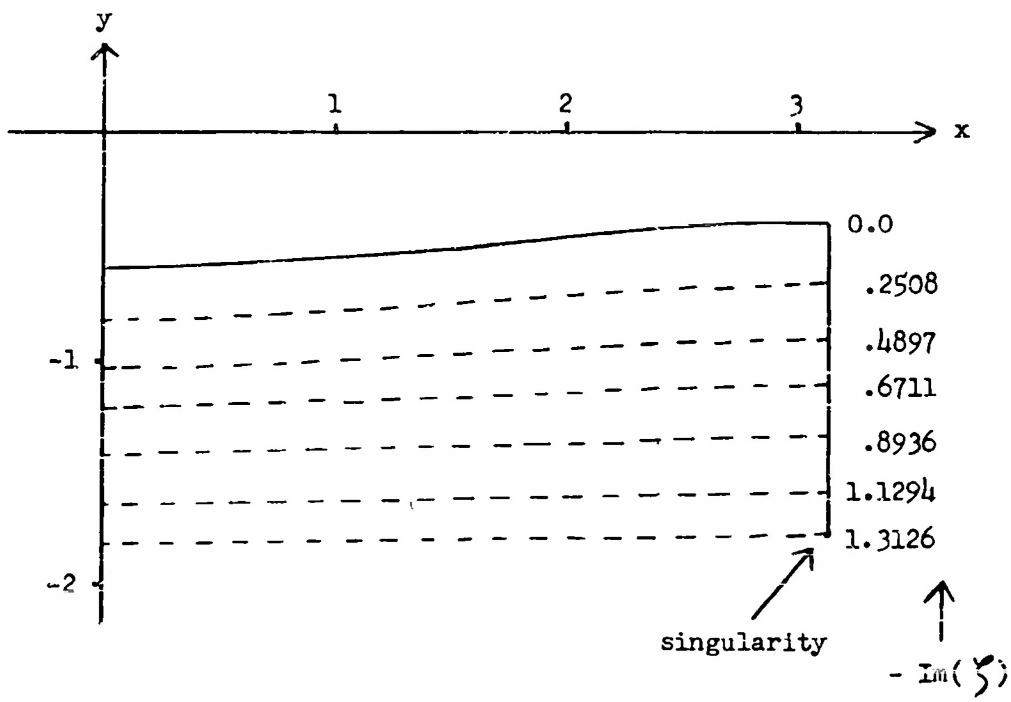


Figure 9

The cell in the z-plane. The dashed lines are not plotted stream lines but serve only to join corresponding known intersections of the stream lines with the cell walls.

Chapter V

We turn our attention to flows that are non-periodic in nature and which have vertical fixed boundaries that are infinite in extent. The vertical fixed boundaries are obtained by forcing the continuation of the free surface to be vertical. If we again consider the differential equation

$$(7) \qquad z' \; (z' + 2\, i\lambda') = \frac{1}{2\lambda},$$

we can determine, to some extent, what type of function λ (ζ) must be chosen. Since we are considering a vertical line, dz is pure imaginary, and since this line is an extension of the free-surface boundary, we are on the real ζ-axis, and dζ is real. Thus, substituting the symbols R_1, R_2, R_3 for real quantities and I_1, I_2 for pure imaginary quantities, equation (7) takes the form

$$I_1 \; (I_2 + 2i \frac{d\lambda}{R_1}) = \frac{1}{2\lambda} \quad ,$$

or

$$R_2 + R_3 d\lambda = \frac{1}{2\lambda}$$

on the vertical fixed boundary. Consequently, if we start from the free surface, where λ and ζ are real, and integrate along the real vertical fixed boundary, the solution λ (ζ) must be real by virtue of the fact that the coefficients of the differential equation for λ are real. This is the only possible choice of λ (ζ), since the solution of equation (7) with prescribed initial data is unique. Thus we must choose real functions for λ (ζ). Furthermore, if we consider the equation

$$(8) \qquad z = - i\lambda + \int \sqrt{\frac{1}{2\lambda} - \lambda'^2} \; d\zeta \; ,$$

we see that the vertical fixed boundary extension of the free surface will

occur when $\frac{1}{2\lambda} - \lambda'^2$ is negative, and will be infinite in extent if either λ or $\int \sqrt{-\frac{1}{2\lambda} + \lambda'^2}\, d\zeta$ becomes infinite for some value of ζ on the extension of the free surface.

From the standpoint of conformal mapping, we see that we must require the lower half of the ζ-plane to map into a portion of the λ-plane that is bounded by part or all of the real λ-axis. That is, the real ζ-axis must map into part or all of the real λ-axis. If there are no singularities on the free surface or its extension, then we must map the half-plane into the half-plane, and the most elementary mapping that accomplishes this is the mapping $\lambda(\zeta) = \zeta$. If we are to have a singularity on the real ζ-axis, it should correspond to $\lambda = 0$, since the differential equation (7) has a singularity at this point. If we wish the entire ζ-axis to map into a part of the real λ-axis, we must have a square root singularity at the origin so that the real negative ζ-axis is doubled back onto the real positive λ-axis. Thus we can use the function $\lambda(\zeta) = \zeta^{1/2}$.

If we set $\lambda = \zeta$, equation (8) takes the simple form

$$z = -i\zeta + \int \sqrt{\frac{1}{2\zeta} - 1}\, d\zeta .$$

A good deal of information about the flow can be obtained directly from this equation without going into the computation. We see that for large positive ζ, the integrand, $\sqrt{\frac{1}{2\zeta} - 1}$, is pure imaginary, so that we are on a vertical fixed boundary. As ζ ranges from 1/2 to 0 we move along the free surface. Since the integral involves the square root of the inverse of ζ, it will be convergent and consequently x will move a finite distance as ζ traverses the interval from 1/2 to 0. During this time, y will decrease from 1/2 to 0. When ζ is negative, we are again

on a fixed boundary. Thus it appears that the flow involves two separate vertical fixed boundaries with a free surface joining them.

As the computation unfolds, we obtain first a semi-fountain in two dimensions. It involves a fixed vertical wall with a flow which moves up the wall, departs from the wall at right angles into a cycloidal free surface which meets another wall tangentially, and finally descends down this latter wall. This flow is illustrated in Figure 10, page 74. If this flow is reflected about the higher wall, we obtain a two-dimensional symmetric fountain. After we have reflected, we may, of course, remove the center wall so that the free surface becomes a complete cycloid. Figure 11 illustrates this reflected flow.

It was mentioned in Chapter I that under certain circumstances we could have a stagnation point on a free surface where the characteristic angle of $\frac{2\pi}{3}$ did not occur. Figure 11 shows an example of such a flow. The center of the cycloidal free surface is a stagnation point on a free surface where the free surface has a continuously turning tangent.

By choosing another branch of the parameter plane, we obtain a flow which occurs in the region of the z-plane which was not included in the above flows. This exterior flow is, in fact, the analytic continuation of the previous interior flow across the free-surface boundary. It comes down uniformly from $+\infty$ along a vertical fixed boundary, and upon meeting an air bubble which is maintained in a slot, separates from the fixed boundary, forming a cycloidal free surface with the air bubble and finally flowing off to $-\infty$ along a vertical fixed boundary displaced horizontally from the original fixed boundary.

This flow is illustrated in Figure 12. Figure 13 is the reflection of this flow with the center wall removed. If we consider ourselves fixed with respect to the outer flow at infinity, we obtain a representation of one fluid mass rising into another. The free surface now becomes the interface between the two fluid masses.

This latter interpretation might be of interest to meteorologists, for one finds such phenomena in the atmosphere. When an air mass, which for this example should be long compared to its width, is heated near the ground, it breaks away from the ground and rises into the cool surrounding air. Of course, in the actual physical occurrence, a wake is formed along the outside edges of the rising air. However, if we idealize the wake by placing partitions in the air which rise with it so that no turbulence is formed on the sides, then the occurrence is described by the example of this chapter. Figure 14 illustrates this flow.

Davis and Taylor [1] have investigated the mechanics of large bubbles rising through extended liquids using an approximate theory. Also Taylor [12] investigated the instability of liquid surfaces when accelerated perpendicular to their planes.

To develop the mathematics of the flow, we return to the equation developed for z,

$$z = -i\zeta + \int \sqrt{\frac{1}{2\zeta} - 1}\, d\zeta ,$$

and we make the substitution $\zeta = \frac{1}{2}\cos^2\theta$. Therefore,

$$d\zeta = -\cos\theta \sin\theta \, d\theta ,$$

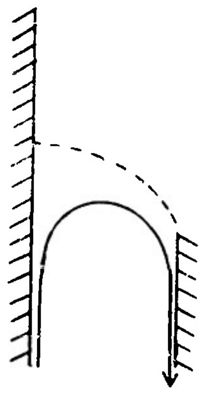


Figure 10

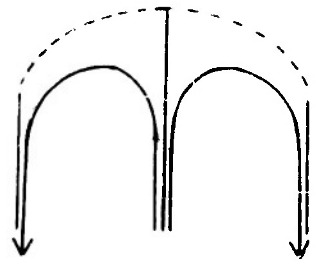


Figure 11

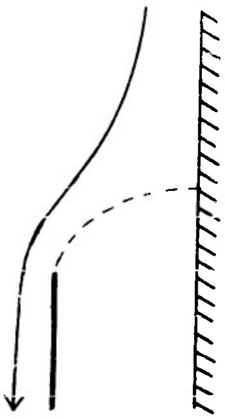


Figure 12

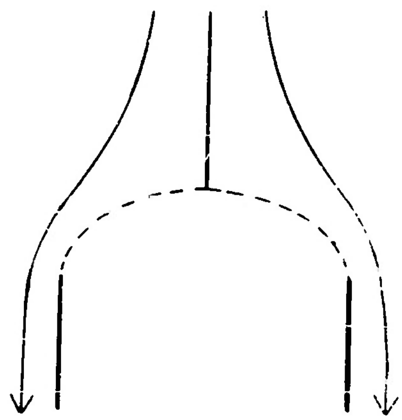


Figure 13

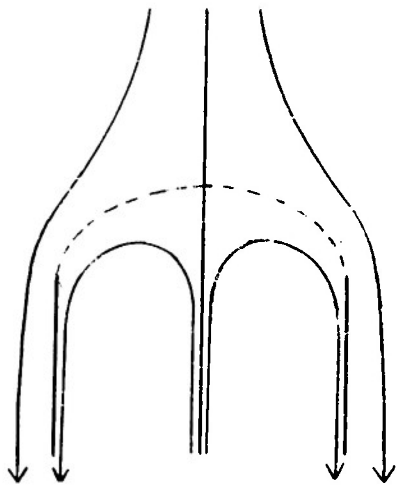


Figure 14

so that

$$z = - i \frac{1}{2} \cos^2 \theta - \int \tan \theta \cos \theta \sin \theta \, d\theta$$
$$= - \frac{i}{2} \cos^2 \theta - \int \sin^2 \theta \, d\theta$$
$$= - \frac{i}{2} \cos^2 \theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4} .$$

This can be written in a neater form if we use the trigonometric identity $2\cos^2 \theta = 1 + \cos 2\theta$. We have for the parametric equations of the flow,

(34) $$\zeta = 1/4(1 + \cos 2\theta),$$

$$z = 1/4(- 2\theta + \sin 2\theta) - \frac{i}{4}(1 + \cos 2\theta).$$

The free surface is obtained by taking θ real so that its parametric equations are

$$x = 1/4(- 2\theta + \sin 2\theta),$$

$$y = - 1/4(1 + \cos 2\theta).$$

These equations are easily recognized as those of a cycloid.

Going into the complex θ-plane, we have, setting $\theta = \xi + i \eta$,

$$\zeta = 1/4[1 + \cos(2 \xi + i 2\eta)]$$

$$= 1/4(1 + \cos 2 \xi \cosh 2\eta - i \sin 2\xi \sinh 2\eta),$$

so that

$$\operatorname{Im}(\zeta) = - 1/4 \sin 2 \xi \sinh 2 \eta .$$

The stream lines are thus obtained for

$$1/4 \sin 2 \xi \sinh 2 \eta = c,$$

or

$$\sin 2 \xi = \frac{k}{\sinh 2 \eta} ; \quad k = 4c.$$

On the free surface ξ and η must satisfy the equation

$$\sin 2\xi \sinh 2\eta = 0.$$

The actual free surface is obtained by taking $\eta = 0$, ξ arbitrary. The continuation of the free surface as a fixed boundary is obtained by taking $2\xi = n\pi$ and η arbitrary. We can discover where the images of these lines are upon computing z for complex values of θ. We have

$$z = -1/4[i + i\cos 2\theta + 2\theta - \sin 2\theta]$$

$$= -1/4[2\theta + i + i(\cos 2\theta + i \sin 2\theta)]$$

$$= -1/4(2\theta + i + i\, e^{i2\theta}).$$

Upon setting $\theta = \xi + i\eta$, this becomes

$$z = -1/4[2\xi + 2i\eta + i + i\, e^{2i(\xi + i\eta)}]$$

$$= -1/4[2\xi + 2i\eta + i + i\, e^{-2\eta}\cos 2\xi - e^{-2\eta}\sin 2\xi],$$

so that we have

$$x = -1/4(2\xi - e^{-2\eta}\sin 2\xi),$$

$$y = -1/4(1 + 2\eta + e^{-2\eta}\cos 2\xi).$$

We see that the fixed boundary continuations of the free surface are of two types. For $\xi = n\pi$ and η arbitrary, the fixed boundaries are lines dropping vertically downwards from the ends of the cycloidal domes. For $\xi = n\frac{\pi}{2}$ the continuations are vertical lines passing through the tops of the cycloidal domes. In any application, only one dome or even just a half-dome is used; however, due to the periodic nature of the equations, there are an infinite number of possible domes spaced along the x-axis.

Figure 15 on page 77 shows the two branches of the parameter plane of θ which yield the internal and external flows. Table 7 contains the results of the computation for the free surface, while Table 8 contains the values of 2ξ and 2η together with the corresponding images of x and y used to find the stream lines $k = \frac{1}{2}$.

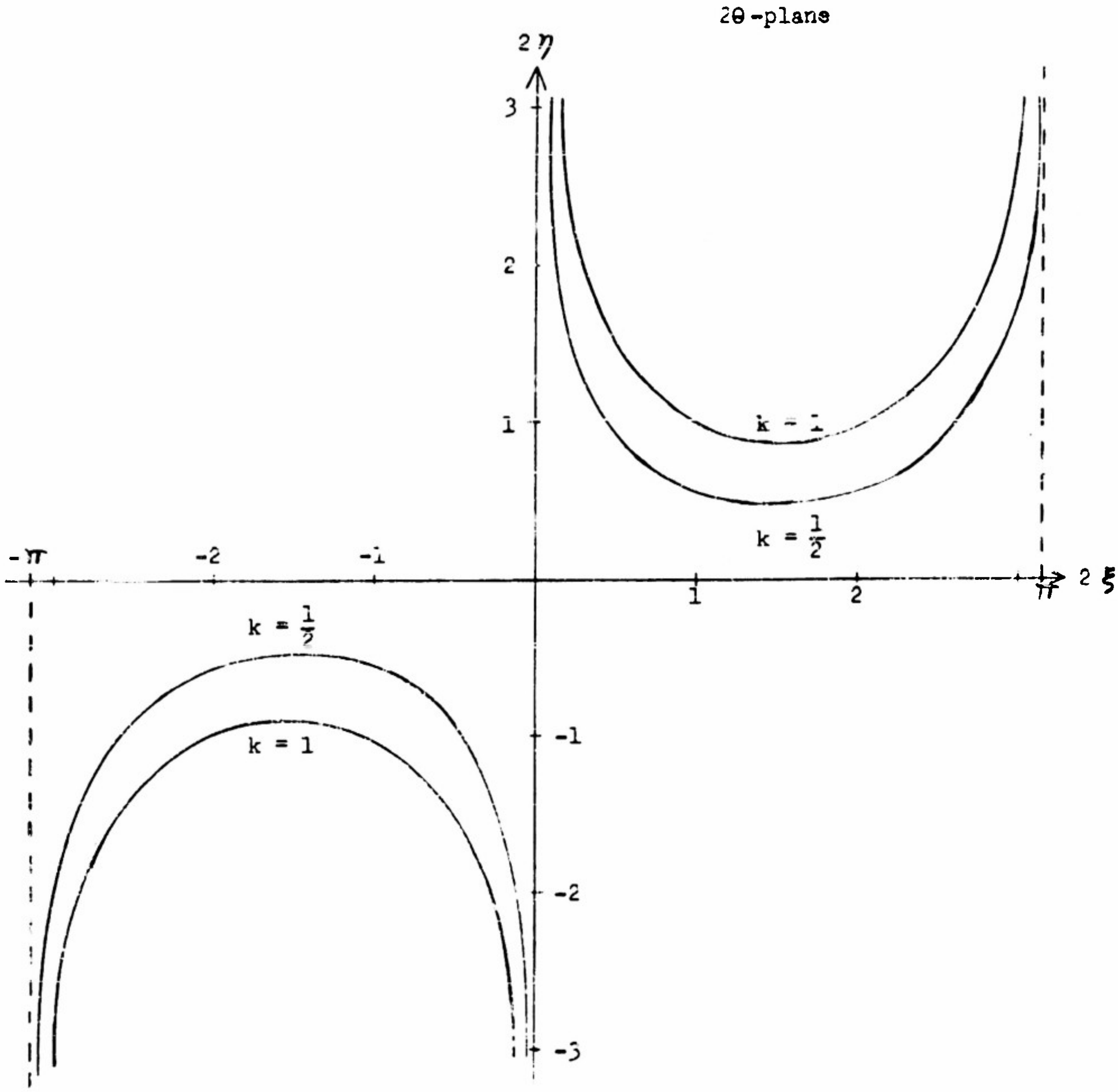


Figure 15

The first quadrant yields the internal flow and
the third quadrant yields the external flow.

We may obtain the velocity expression from the relation

$$\vec{q} = \left(\overline{\frac{d\zeta}{dz}}\right) = \frac{\left(\overline{\frac{d\zeta}{d\theta}}\right)}{\left(\overline{\frac{dz}{d\theta}}\right)} .$$

From equation (34) we have

$$\frac{d\zeta}{d\theta} = -\frac{1}{2}\sin 2\theta,$$

$$\frac{dz}{d\theta} = -\frac{1}{2} + \frac{1}{2}\cos 2\theta + \frac{i}{2}\sin 2\theta,$$

whence

$$\vec{q} = \frac{-\sin 2\bar{\theta}}{\cos 2\bar{\theta} - 1 - i\sin 2\bar{\theta}} = \frac{\sin 2\bar{\theta}\,(1 - \cos 2\theta) - i\sin^2 2\bar{\theta}}{2 - 2\cos 2\bar{\theta}} .$$

If we use the identity $\sin^2 2\bar{\theta} = 1 - \cos^2 2\bar{\theta} = (1 - \cos 2\bar{\theta})(1 + \cos 2\bar{\theta})$, this simplifies to

$$\vec{q} = \frac{1}{2}\left[\sin 2\bar{\theta} - i(1 + \cos 2\bar{\theta})\right] = -\frac{i}{2}\left(1 + e^{i\,2\bar{\theta}}\right).$$

Finally, inserting for $2\bar{\theta}$ the quantity $2\xi - i2\eta$, we have

$$\vec{q} = \frac{1}{2}e^{2\eta}\sin 2\xi - \frac{i}{2}\left(1 + e^{2\eta}\cos 2\xi\right).$$

It can be seen from this expression that the arrows shown on the stream lines in Figures 10 and 12 are correct.

As was mentioned in the beginning of this chapter, another possible choice for the arbitrary function $\lambda(\zeta)$ is the quantity $\zeta^{1/2}$. With this function, we obtain a relation which represents the flow past a semi-infinite, thin, vertical obstacle. On the down stream or right-hand side of the obstacle, a free surface is formed which leaves the obstacle tangentially, but quickly turns in the direction of flow. On the upstream side of the obstacle, the flow parts at a stagnation point occuring a short

way up the obstacle with the upper part of the flow passing off to infinity in the vertical direction. This example, which is shown in Figure 16 on page 80, is not physically significant in the large, for the free surface continues to drop as we go away from the obstacle, so that very large velocities occur at points far removed from the origin. Furthermore, on the upstream side of the obstruction we must either impose a fixed boundary, or allow negative pressures and an infinite piling-up of the water in front of the obstacle. If, however, we consider only the local properties of the mathematical flow, we can expect to obtain a fairly good representation of an actual physical occurrence. In this example, the best physical picture seems to be the high-speed skimming of the ocean surface by a long, thin, vertical obstacle. Our coordinate system, however, has been chosen so that, mathematically, we are riding with the obstacle and the ocean is flowing past us.

In this example it is more convenient to join z and ζ through the parameter λ. From $\lambda = \zeta^{1/2}$, we have $\zeta = \lambda^2$, so that

$$d\zeta = 2\lambda \, d\lambda ,$$

and

$$\frac{d\lambda}{d\zeta} = \frac{1}{2\lambda},$$

whence

$$\lambda'^2 = \left(\frac{1}{2\lambda}\right)^2 .$$

We substitute these quantities into equation (8) and obtain

$$\begin{aligned} z &= -i\lambda + \int \sqrt{\frac{1}{2\lambda} - \left(\frac{1}{2\lambda}\right)^2}\, 2\lambda \, d\lambda \\ &= -i\lambda + \int \sqrt{2\lambda - 1}\, d\lambda \\ &= -i\lambda + (2\lambda - 1)^{3/2}. \end{aligned}$$

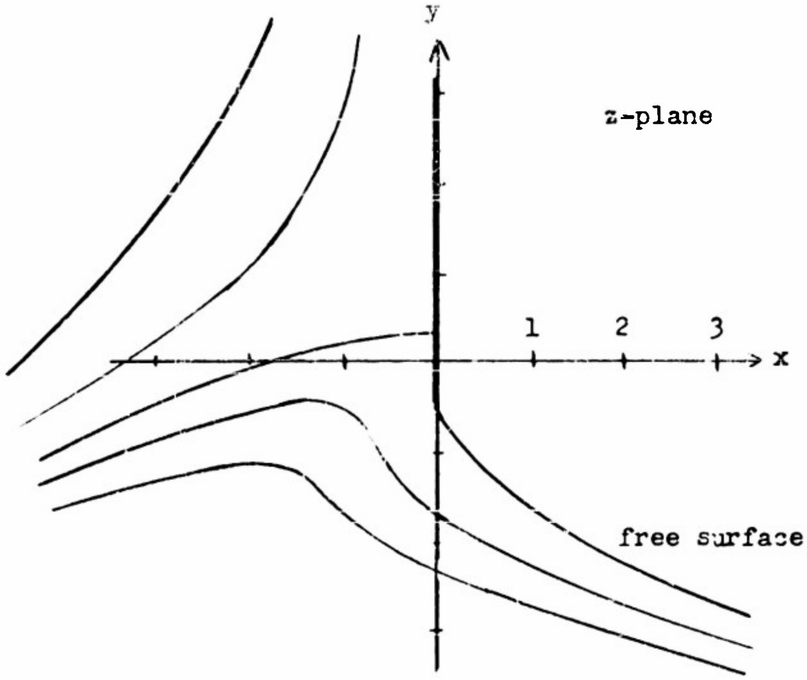


Figure 16

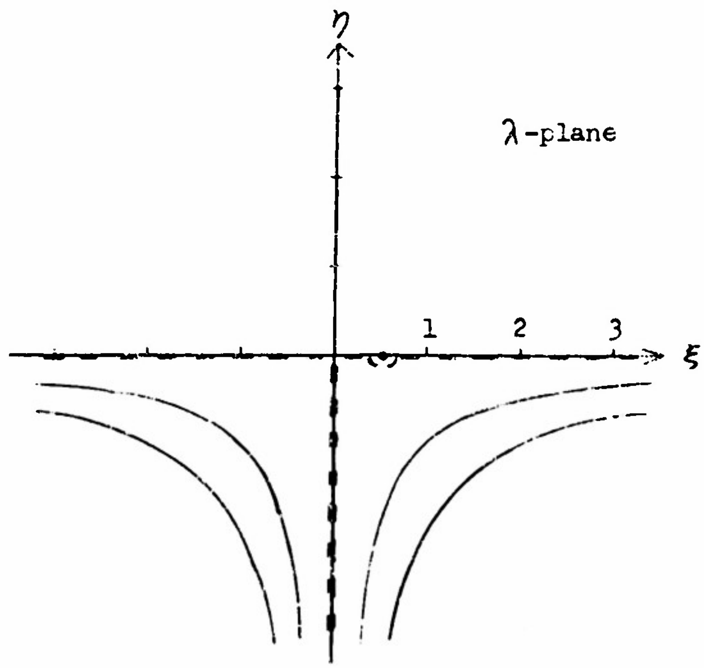


Figure 17

The fourth quadrant yields the lower part of the flow.

The behavior of the free stream line and the fixed boundary extension of the free stream line are now apparent. For $\lambda > 1/2$ we are on the free surface which extends to infinity toward the right. For $\lambda < 1/2$ we are on the fixed vertical boundary, which also extends to infinity, but does so vertically upwards. The point $\lambda = 0$ is a branch point of the mapping and we can expect the image of this point on the vertical fixed boundary to be a stagnation point where the stream lines separate.

The exact behavior of the flow, however, can be determined only through calculation. We have $\zeta = \lambda^2$, so that if we let $\lambda = \xi + i\eta$, we obtain, for the images of the stream lines, the relation

$$\operatorname{Im}(\zeta) = 2\xi\eta = \text{constant}.$$

These are, of course, equilateral hyperbolas with the coordinate axes as asymptotes, as illustrated in Figure 17. If we set the constant equal to zero, we find that the free surface, together with all its possible extensions, is the image of the ξ- and η-axes.

The most interesting part of the flow is obtained by considering the fourth quadrant of the λ-plane. As has already been pointed out, the actual free surface is the image of that part of the real λ-axis for which $\xi \geq 1/2$, since there the free-surface condition

$$\frac{1}{2\lambda} - \lambda'^2 \geq 0, \quad \text{or} \quad \frac{1}{2\xi} \geq \frac{1}{(2\xi)^2}$$

is satisfied. The free surface is hence given parametrically by the equations

$$x = 1/3\,(2\xi - 1)^{3/2},$$
$$y = -\xi, \quad \xi \geq 1/2 .$$

Thus the free surface resembles the curve $y = x^{2/3}$.

In order to stay within the region of flow (the fourth quadrant), we must pass below the branch point at $\xi = 1/2$; consequently, the argument of $2\lambda - 1$ will change from zero to $-\pi$. Thus the argument of $(2\lambda - 1)^{3/2}$ changes from zero to $-3\pi/2$, so that

$$(2\lambda - 1)^{3/2} = i\,(1 - 2\lambda)^{3/2},$$

for $0 \leq \lambda \leq 1/2$. This part of the extension of the free surface as a fixed boundary is therefore the segment of the y-axis from $y = -1/2$ to $y = 1/3$.

The point $\lambda = 0$ is a branch point in the mapping. If we were to continue in the direction of the negative real λ-axis, we would continue, in the z-plane, to move up the vertical fixed boundary. However, to stay in the branch of the mapping within the fourth quadrant of the λ-plane, we must turn at this point and move down the negative η-axis. We set $\lambda = it^2$ and obtain

$$z = -t^2 + \frac{i}{3}(1 + 2it^2)^{3/2}.$$

This is the stream line that meets the vertical fixed boundary at a stagnation point. As this particular line is of interest, we will obtain its parametric equations. Let us call it the "zero" stream line.

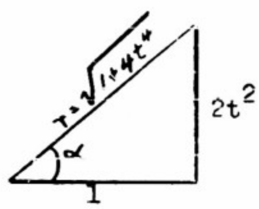
The argument of $1 + 2it^2 = r\,e^{i\alpha}$ is between zero and $\pi/2$ and is given by $\alpha = \arctan 2t^2$. In terms of α, r and t, we may write

$$z = -t^2 + \frac{i}{3} r^{3/2} \left(\cos \tfrac{3}{2}\alpha + i \sin \tfrac{3}{2}\alpha\right) ,$$

whence

$$x = -t^2 - \frac{1}{3} r^{3/2} \sin \tfrac{3}{2}\alpha ,$$

$$y = \frac{1}{3} r^{3/2} \cos \tfrac{3}{2}\alpha .$$



Employing the identities

$$\cos\frac{3}{2}\alpha = (2\cos\alpha - 1)\left(\frac{1+\cos\alpha}{2}\right)^{1/2}, \quad (0 \le \alpha \le \pi),$$
$$\sin\frac{3}{2}\alpha = (2\cos\alpha + 1)\left(\frac{1-\cos\alpha}{2}\right)^{1/2}, \quad (0 \le \alpha \le \pi)$$

and the relations

$$\cos\alpha = \frac{1}{r}, \quad t^2 = \frac{1}{2}(r^2 - 1)^{1/2}, \quad (r \ge 1)$$

we obtain

$$x = -\frac{1}{2}(r^2-1)^{1/2} - \frac{1}{3}r^{3/2}\left(\frac{2}{r}+1\right)\left(\frac{1-\frac{1}{r}}{2}\right)^{1/2},$$
$$y = \frac{1}{3}r^{3/2}\left(\frac{2}{r}-1\right)\left(\frac{1+\frac{1}{r}}{2}\right)^{1/2}, \quad (r \ge 1)$$

or, upon simplification,

$$x = -\left(\frac{r-1}{2}\right)^{1/2}\left[\left(\frac{1+r}{2}\right)^{1/2} + \frac{1}{3}(2+r)\right],$$
$$y = \frac{1}{3}(2-r)\left(\frac{1+r}{2}\right)^{1/2}, \quad (r \ge 1).$$

The derivative assumes the simple form

$$\frac{dy}{dx} = \frac{\sqrt{r-1}}{\sqrt{2} + \sqrt{r+1}}$$

This further extension of the free surface accordingly leaves the y-axis at right angles in the negative x direction, turns immediately downwards and finally has asymptotically the slope +1.

If we look for the flow above this "zero" stream line we must consider the third quadrant in the λ-plane. The images of the stream lines here correspond to choosing positive constants for the imaginary part of ζ; consequently, we will soon run into negative pressures if we stray far from the "zero" stream line.

Considering again the lines Im(ζ) = 0, but dwelling now in the third quadrant of the λ-plane, we first retrace our steps up the η-axis. In the z-plane we are of course coming back up th "zero" stream line, only now we are on top of it. When we reach t[illegible] origin in the λ-plane, we turn and run out the negative real axis. Thus $\eta = 0, \xi = -p^2$ and we have

$$z = ip^2 + \frac{1}{3} i (1 + 2p^2)^{3/2}.$$

This is the part of the y-axis above y = 1/3.

The picture of the flow now becomes clear. The fluid flows up from the left at an angle of approximately 45°, strikes a plate obstacle which separates the flow into a part which goes upward, and a part which flows downward and soon separates from the plate into a free-surface wake. The stagnation point where the flow divides is the point where the "zero" stream line meets the y-axis.

Turning to the velocity, we use

$$d\zeta = 2\lambda \, d\lambda ,$$

$$dz = (-i + \sqrt{2\lambda - 1}) \, d\lambda$$

to obtain

$$q = q_1 + iq_2 = \overline{\left(\frac{d\zeta}{dz}\right)}$$

$$= \frac{2\bar{\lambda}}{i + \sqrt{2\bar{\lambda} - 1}}$$

$$= -i + \sqrt{2\bar{\lambda} - 1}.$$

On the free surface,

$$q_1 = \sqrt{2\xi - 1}, \qquad \xi \geq 1/2.$$

$$q_2 = -1.$$

On the vertical segment from $y = -1/2$ to $y = 1/3$,

$$q_1 = 0,$$

$$q_2 = -1 + \sqrt{1 - 2\xi}\ , \qquad 0 \leq \xi \leq 1/2.$$

Finally along the "zero" stream line we have

$$q_1 = \left(\frac{r - 1}{2}\right)^{1/2},$$

$$q_2 = -1 + \left(\frac{r + 1}{2}\right)^{1/2},$$

where this r is equal to the r used in the parametric representation of the "zero" stream line. Along the vertical segment from $y = 1/3$ to $y = +\infty$ we can use the relations developed for the lower segment, except that now ξ would range from 0 to $-\infty$.

Table 1

Points on the free surface for the trochoidal flow.

$$x = -\xi + \frac{1}{2\sqrt{2}} \sin\xi \; ; \quad y = -\frac{9}{16} + \frac{1}{2\sqrt{2}} \cos\xi$$

ξ	$-x$	$-y$
0	0	.210
.10	.065	.213
.20	.129	.217
.50	.330	.252
.75	.510	.305
1.00	.703	.372
1.25	.914	.450
$\pi/2$	1.22	.563
1.75	1.40	.627
2.00	1.68	.711
2.25	1.97	.786
2.50	2.29	.846
2.75	2.62	.888
2.90	2.82	.906
3.00	2.95	.913
π	3.14	.917
3.30	3.36	.913
3.50	3.62	.895
3.75	3.95	.853
4.00	4.27	.793
4.50	4.85	.637
5.00	5.34	.460
5.50	5.75	.312
2π	6.28	.210

Table 2

Points on the stream line k = .25 for the trochoidal flow.

$$\xi = \cos^{-1}\left\{\frac{1.125\eta - .25}{.707\sinh\eta}\right\};\quad x = -\xi + \frac{1}{2\sqrt{2}}e^{\eta}\sin\xi;\quad y = -\eta - \frac{9}{16} + \frac{1}{2\sqrt{2}}e^{\eta}\cos\xi .$$

η	ξ	$-x$	$-y$
.1363	π	π	1.104
.1365	3.05	3.014	1.103
.137	2.98	2.92	1.100
.14	2.76	2.61	1.080
.15	2.44	2.18	1.026
.17	2.08	1.71	.936
.2	1.745	1.32	.839
.3	1.15	.714	.669
.4	.81	.42	.599
.5	.56	.25	.568
.6	.335	.123	.554
.65	.215	.064	.551
.67	.16	.050	.550
.69	.07	.021	.549
.695	0	0	.549

Table 3

Values of η_1 and η_2 together with Δy, 2b, % smoothing and wave height; wave length ratio.

$$\eta_1 = -\log b; \ \eta_2 + \frac{2b}{b^2 + 1} \sinh \eta_2 = \eta_1 + \frac{b^2 - 1}{b^2 + 1}; \ \Delta y = be^{\eta_2} + \eta_2 + 1 - \eta_1$$

$$\% \text{ smoothing} = \frac{2b - \Delta y}{2b} 100; \quad \text{wave length} = 2\pi.$$

b	η_1	η_2	Δy	2b	% smoothing	ratio
.4	.916	.1136	.6457	.8	19.3	1:8
.3	1.204	.2374	.4138	.6	31.0	1:10
.2	1.609	.490	.2075	.4	48.1	1:16
.15	1.897	.714	.1233	.3	59.0	1:21
.1	2.303	1.07	.0585	.2	70	1:31
.05	2.996	1.73	.0160	.1	84	1:62
.025	3.6889	2.4135	.00393	.05	92	1:125

Table 4

Depth

$-y_b$ = $-y$ value of bottom below trough = $\eta_2 + \frac{b^2+1}{2} + be^{\eta_2}$

$-y_s$ = $-y$ value of surface at trough = $\frac{b^2+1}{2} + b$

depth = $y_s - y_b$

b	$-y_b$	$-y_s$	depth
.4	1.1417	.98	.1617
.3	1.1628	.845	.3178
.2	1.336	.72	.616
.15	1.532	.661	.871
.1	1.867	.605	1.262
.05	2.513	.551	1.962
.025	3.193	.525	2.668

Table 5

Intersections of the stream lines with the walls of the cell.

$\lambda \leq .4$	$\lambda \geq .6$	$-\mathrm{Im}(\zeta)$	-y trough	-y crest	Δ y
.4	.6	0	.6	.4	.2
.395	.60219	.2508	.831	.677	.154
.38	.6080	.4897	1.049	.936	.113
.36	.6158	.6711	1.223	1.128	.095
.34	.6224	.7972	1.343	1.260	.083
.32	.6280	.8936	1.431	1.360	.071
.30	.6335	.9706	1.515	1.440	.075
.28	.6385	1.0335	1.593	1.505	.088
.26	.6420	1.0864	1.624	1.563	.061
.24	.6450	1.1294	1.653	1.604	.049
.20	.6510	1.1971	1.728	1.673	.055
.16	.6560	1.2446	1.783	1.722	.061
.12	.6584	1.2772	1.798	1.755	.043
.08	.6602	1.2979	1.827	1.776	.051
.04	.6607	1.3092	1.832	1.788	.044
0	.6611	1.3126	1.835	1.791	.044

Table 6

Pairs of values on the free surface.

x	-y
0	.6
.348	.595
.702	.58
1.004	.56
1.247	.54
1.462	.52
1.665	.50
1.863	.48
2.066	.46
2.285	.44
2.547	.42
2.842	.405
3.127	.4

Table 7

Points on a cycloid.

$$4x = -2\theta + \sin 2\theta$$

$$4y = -1 - \cos 2\theta$$

$$0 \leq 2\theta \leq \pi$$

2θ	$-4x$	$-4y$
0	0	2.0
.3	.004	1.96
.5	.02	1.88
1.0	.16	1.54
1.57	.57	1.0
2.14	1.30	.46
2.64	2.16	.12
2.84	2.54	.04
3.14	3.14	0

Table 8

2ξ and 2η such that $\sin 2\xi \sinh 2\eta = \frac{1}{2}$ and corresponding $4x$ and $4y$.

$$4x = -2\xi + e^{-2\eta} \sin 2\xi \; ; \quad 4y = -1 - 2\eta - e^{-2\eta} \cos 2\xi$$

2ξ	2η	$4x$	$4y$
3.14	$+\infty$	-3.14	$-\infty$
3.04	2.31	-3.03	-3.21
2.84	1.28	-2.76	-2.01
2.64	.91	-2.45	-1.56
2.14	.56	-1.66	-1.25
1.67	.49	-1.06	-1.43
1.57	.48	- .95	-1.48
1.47	.49	- .86	-1.55
1.0	.56	- .52	-1.87
.5	.91	- .31	-2.26
.3	1.28	- .22	-2.55
.1	2.31	- .09	-3.41
-.1	-2.31	- .91	-8.74
-.3	-1.28	- .78	-3.18
-.5	-.91	- .69	-2.27
- 1.0	-.56	- .47	-1.39
-1.47	-.49	- .15	-.67
-1.57	-.48	- .05	-.52
-1.67	-.49	.05	-.35
-2.14	-.56	.67	.51
-2.64	-.91	1.45	2.09
-2.84	-1.28	1.76	3.74
-3.04	-2.31	2.03	11.36

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